Interaction of light with gravitational waves

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Abstract

Geodesics in plane background gravitational wave is solved in order to understand how photon moves with gravitational waves. Coupling of electromagnetic waves, which is another interpretation of light, and curvature of spacetime is also calculated using perturbation method but it is still controversial about its correctness. Other methods of understanding the interaction between light and gravitational waves are discussed. Note that this is only an intermediate report of the project.

1 Introduction

In 2016, LIGO detected the signal of gravitational wave (GW), which was produced by binary black holes merger(Abbott et al. 2016). This was the first time humans ever detected GW directly. In 2017, LIGO detected another signal of GW produced by binary neutron stars merger (Abbott et al., 2017). What is more interesting is the detection of gamma ray burst signal from the same event, which ignites the field of multi-messenger astronomy. Possible interaction between gamma ray burst and GW from the same source is one of the motivation of this project. This project is more important because of the use of pulsar timing array (PTA) for GW detection. This method is important because it can detect GW in the frequency between 10^{-9} and 10^{-7}Hz (Moore, Taylor & Gair, 2015) which can be hardly detected using interferometer like LIGO. Before PTA works, we need to understand very well how GW affect the propagation of light.

In this report, the geodesics of photon in plane GW background is solved in order to have a brief feeling about the interaction between photon and GW. For your interest, a controversial method of electromagnetic field-GW curvature coupling is also worked out. I did not guarantee the correctness of any physical meaning of the result. The result would be interesting if it is correct to certain extent, so I want to show the result anyway. Most of the formulations can be found in [4].

2 Gravitational waves

For a GW traveling in flat spacetime, the metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.1}$$

,where $\eta_{\mu\nu} = diag(-1,1,1,1)$ is the Minkowski metric and $h_{\mu\nu}$ is the perturbation of spacetime due to GW. Plane GW of the following form is considered

$$h_{11} = h_+ \cos[k_+(z-t)+\phi_1], h_{22} = -h_+ \cos[k_+(z-t)+\phi_1] \tag{2.2}$$

$$h_{12} = h_{21} = h_\times \cos[k_\times(z-t)+\phi_2] \tag{2.3}$$

, $h_{\mu\nu}$ is zero otherwise. GW of this form represents a z-propagating GW with two polarization mixed. Under the scheme of linearised general relativity, GWs obey the rule of superposition. Therefore, it is valid that the two polarizations are with different amplitude, frequency and phase. The above assumptions of GW are only valid when dealing with GW far away from the source, which means h_+ and $h_\times$ are extremely small. For instance, on Earth, we usually detect GW of amplitude $\sim 10^{-21}$.

3 Formulations

This section provides all ingredients needed to solve the null geodesic for moving photons.

3.1 Geodesic equations

Motion of a particle in a spacetime always follows the geodesic equations. The geodesic equations is given by

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0 \tag{3.1}$$

Due to symmetry, geodesic equations of t and z are the same, which leads to the following equation

$$\frac{d^2}{d\lambda^2}(z-t) = 0 \tag{3.2}$$

, where λ is the affine parameter. Therefore, $z - t = A\lambda + B$. The affine parameter can be freely transformed by $\lambda = a\lambda_{new} + b$, so we obtain

$$z - t = -\lambda \tag{3.3}$$

3.2 Null geodesics

Besides geodesic equations, photons also obey the null geodesic equation

$$g_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = 0 \tag{3.4}$$

, which is

$$-\dot{t^2} + (1 + h_+\cos(k_+\lambda - \phi_1))\dot{x^2} + (1 - h_+\cos(k_+\lambda - \phi_1))\dot{y^2} + 2h_\times\cos(k_\times\lambda - \phi_2)\dot{x}\dot{y} + \dot{z^2} = 0 \tag{3.5}$$

3.3 Killing vectors

The metric is invariant under changes in x or y coordinates and hence inhibits conservation. The conserved quantities can be easily obtained through the usage of killing vectors. Killing vectors of the given metric is

$$K_\mu = g_{\mu 1}, K'_\mu = g_{\mu 2} \tag{3.6}$$

, so $K_\mu x^\mu$ is a conserved quantities. Therefore

$$(1 + h_+ \cos(k_+\lambda - \phi_1))\dot{x} + h_\times \cos(k_\times\lambda - \phi_2)\dot{y} = c_x \tag{3.7}$$

$$h_\times \cos(k_\times\lambda - \phi_2)\dot{x} + (1 - h_+ \cos(k_+\lambda - \phi_1)\dot{y} = c_y \tag{3.8}$$

,where c_x and c_y are two conserved quantities describing the photon. $c_x \approx \dot{x}$ and $c_y \approx \dot{y}$ when we ignore the effect of GW. That means c_x and c_y are closely related to the speed of photon in x and y direction respectively.

4 Exact equations of motion

In section 3, all formulas needed are introduced to obtain the equations of motion of a photon in plane GW. Eq.(4.1) & (4.2) can be obtained from Eq.(3.7) & (3.8). Substitute Eq.(3.3), (3.7) & (3.8) into Eq.(3.5) one gets Eq.(4.3) & (4.4).

$$\dot{x} = \frac{c_x - c_x h_+ \cos(k_+\lambda - \phi_1) - c_y h_\times \cos(k_\times\lambda - \phi_2)}{1 - h_+^2 \cos^2(k_+\lambda - \phi_1) - h_\times^2 \cos^2(k_\times\lambda - \phi_2)} \tag{4.1}$$

$$\dot{y} = \frac{c_y + c_y h_+ \cos(k_+\lambda - \phi_1) - c_x h_\times \cos(k_\times\lambda - \phi_2)}{1 - h_+^2 \cos^2(k_+\lambda - \phi_1) - h_\times^2 \cos^2(k_\times\lambda - \phi_2)} \tag{4.2}$$

$$\dot{z} = \frac{1}{2}(c_x\dot{x} + c_y\dot{y}) - \frac{1}{2} \tag{4.3}$$

$$\dot{t} = \frac{1}{2}(c_x\dot{x} + c_y\dot{y}) + \frac{1}{2} \tag{4.4}$$

5 Preliminary results

5.1 General solutions

It is difficult to integrate Eq.(4.1) $\sim$ (4.4) directly. Nevertheless, magnitude of the GW is very weak (e.g. $h_+, h_\times \sim 10^{-21}$), so Eq.(4.1) and Eq.(4.2) can be written as

$$\dot{x} = (c_x - c_x h_+ \cos(k_+\lambda - \phi_1) - c_y h_\times \cos(k_\times\lambda - \phi_2))(1 + h_+^2 \cos^2(k_+\lambda - \phi_1) + h_\times^2 \cos^2(k_\times\lambda - \phi_2) + \mathcal{O}(\max(h_+^4, h_\times^4))) \tag{5.1}$$

$$\dot{y} = (c_y + c_y h_+ \cos(k_+\lambda - \phi_1) - c_x h_\times \cos(k_\times\lambda - \phi_2))(1 + h_+^2 \cos^2(k_+\lambda - \phi_1) + h_\times^2 \cos^2(k_\times\lambda - \phi_2) + \mathcal{O}(\max(h_+^4, h_\times^4))) \tag{5.2}$$

Therefore,

$$\begin{aligned} x(\lambda) = \; & \underbrace{c_x\lambda}_{\text{0th order}} \underbrace{- \frac{c_x h_+}{k_+} \sin(k_+\lambda - \phi_1) - \frac{c_y h_\times}{k_\times} \sin(k_\times\lambda - \phi_2)}_{\text{1st order}} \\ & \underbrace{+ c_x(\frac{h_+^2 + h_\times^2}{2}\lambda + \frac{h_+^2}{4k_+} \sin(2k_+\lambda - 2\phi_1) + \frac{h_\times^2}{4k_\times} \sin(2k_\times\lambda - 2\phi_2))}_{\text{2nd order}} \underbrace{+ \mathcal{O}(\max(h_+^3, h_\times^3))}_{\text{3rd order}} \end{aligned} \tag{5.3}$$

$$\begin{aligned} y(\lambda) = \; & \underbrace{c_y\lambda}_{\text{0th order}} \underbrace{+ \frac{c_y h_+}{k_+} \sin(k_+\lambda - \phi_1) - \frac{c_x h_\times}{k_\times} \sin(k_\times\lambda - \phi_2)}_{\text{1st order}} \\ & \underbrace{+ c_y(\frac{h_+^2 + h_\times^2}{2}\lambda + \frac{h_+^2}{4k_+} \sin(2k_+\lambda - \phi_1) + \frac{h_\times^2}{4k_\times} \sin(2k_\times\lambda - 2\phi_2))}_{\text{2nd order}} \underbrace{+ \mathcal{O}(\max(h_+^3, h_\times^3))}_{\text{3rd order}} \end{aligned} \tag{5.4}$$

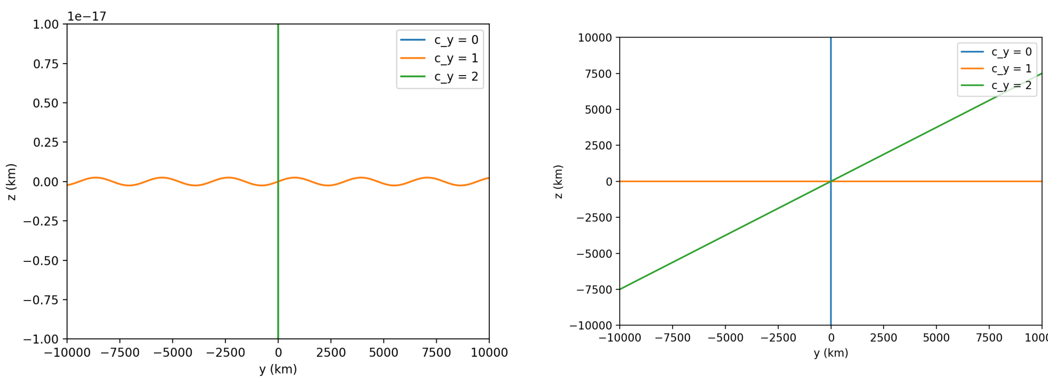
$$z(\lambda) = \frac{1}{2}(c_x x + c_y y) - \frac{\lambda}{2} \tag{5.5}$$

$$t(\lambda) = \frac{1}{2}(c_x x + c_y y) + \frac{\lambda}{2} \tag{5.6}$$

I keep the terms to the 2nd order because it shows some special feature. For example, $\frac{a^2+b^2}{2}\lambda$ represents the photon constantly push by GW, while $\sin 2k_+\lambda$ and $\sin(2k_\times\lambda + 2\phi)$ illustrates other periodicities of the solutions. Higher order terms can be worked out whenever necessary.

5.2 Plus polarization

Set $h_\times = 0$ so the GW is only composited of plus polarization. Furthermore, set $h_+ = 10^{-21}, k_+ = 0.002km^{-1}$ and $\phi_1 = 0$ This simply model gives us a primary feeling of the effect of GW on a moving photon. For convenience, let $c_x = 0$ so that the geodesic of photons is confined on y-z plane. Fig.1 shows the geodesics of different values of c_y. First look at $c_y = 1$, it's path is affected by the gravitational waves so it is shaking while propagating, but the effect is very tiny as you can see from the scale. The $c_y = 1$ curve is actually slowly moving upwards (z-direction), but the effect is too small to see here. For $c_y = 2$, it looks like a straight line just because of the scaling problem. I am unable to show the shaking of it, but from Eq.(18) you know it is fluctuating like $c_y = 1$. For $c_y = 0$, it is a completely straight line. It is blocked in Fig.1a, but you can see it in Fig.1b. This means if the light and GW propagate in the same direction, the light will not be affected, and the speed is c. The geodesic looks as if it is a straight line if someone sees it from far away (as shown in Fig. 1b).



(a) Zoom in

(b) Zoom out

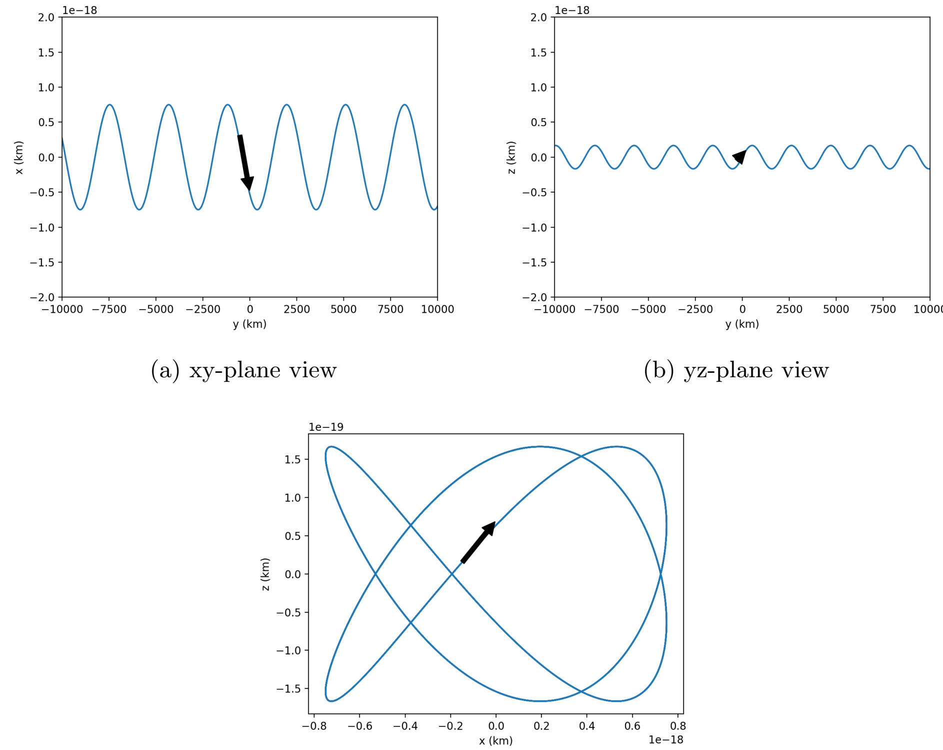
Figure 1: Geodesics of photons at different values of c_y, while $c_x = 0$. GW is propagating in z-direction, where $k_+ = 0.002km^{-1}, h_+ = 10^{-21}$

5.3 Mixed polarization

Put $c_x = 0, c_y = 1$,

$$x(\lambda) = -\frac{h_\times}{k_\times}\sin(k_\times\lambda - \phi_2) \tag{5.7}$$

$$y(\lambda) = \lambda + \frac{h_+}{k_+}\sin(k_+\lambda - \phi_1) + \frac{h_+^2 + h_\times^2}{2}\lambda + \frac{h_+^2}{4k_+}\sin(2k_+\lambda - 2\phi_1) + \frac{h_\times^2}{4k_\times}\sin(2k_\times\lambda - 2\phi_2) \tag{5.8}$$



(a) xy-plane view

(b) yz-plane view

(c) xz-plane view

Figure 2: Motion of photon on xz-plane, where $a = 1.5 \times 10^{-21}, b = 4 \times 10^{-21}, k_+ = 0.003km^{-1}, k_\times = 0.002km^{-1}, \phi_1 = 0, \phi_2 = \pi/4, c_x = 0, c_y = 1$

$$z(\lambda) = \frac{1}{2}(\frac{h_+}{k_+}\sin(k_+\lambda - \phi_1) + \frac{h_+^2 + h_\times^2}{2}\lambda + \frac{h_+^2}{4k_+}\sin(2k_+\lambda - 2\phi_1) + \frac{h_\times^2}{4k_\times}\sin(2k_\times\lambda - 2\phi_2)) \tag{5.9}$$

Unlike single polarization situation, the photon can no longer be confined on y-z plane even I set $c_x = 0$. Photon is mainly pointing to the y-direction as you can see from the equations. Meanwhile, the photon is fluctuating both in x and z direction. Figure 2 shows the fluctuation of photon on xz-plane. In Figure 2, photon's motion looks periodic, it is because the first order of Eq.(46) is dominating. The period is around 133 seconds under the given parameters. The displacement is unmeasurable but only due to the magnitude of h_+ and $h_\times$. Imaging a light source near a GW source (compare to Earth), so h_+ and $h_\times$ can be up to 10^{-8}. Put this value back to the solution then the displacement becomes as large as 1cm. Although the validity of this simply model in a nearer field is in question, it hints that in some cases the effect of GW on a propagating photon is non-negligible.

5.4 Remark

If one is interested in mixed polarization with same wavelengths and phases, but only differ by the magnitude of the mode, which means $h_+ \neq h_\times$, it is actually same as a single polarization as shown in the following. The perturbative metric $h_{\mu\nu}$ is now

$$h_{11} = h_+ \cos k(z-t), h_{22} = -h_+ \cos k(z-t), h_{12} = h_{21} = h_\times \cos k(z-t) \tag{5.10}$$

, otherwise zero. By performing a rotation transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \tag{5.11}$$

, where $\tan 2\theta = b/a$. Then, the metric becomes

$$ds^2 = -dt^2 + (1+(a\cos 2\theta + b\sin 2\theta)\cos k(z-t))dX^2 + (1-(a\cos 2\theta + b\sin 2\theta)\cos k(z-t))dY^2 + dz^2 \tag{5.12}$$

, which means in this case, it is still a single polarization GW.

6 Massive particles

It is easy to generalize the calculation to massive particles. Instead of Eq.(3.4), massive particle obey the following equation

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1 \tag{6.1}$$

, where the affine parameter is now the proper time of the particle. Affine parameter is well defined, so Eq.(3.3) has to be modified

$$z - t = -c_z\tau + Z = -\lambda \tag{6.2}$$

c_z is related to particle speed in z direction. Keep λ as the parameter for the differential equations. The remaining maths are very similar. The equations of motion are

$$\dot{x} = \frac{1}{c_z}\frac{c_x - c_x a\cos k_+\lambda - c_y b\cos(k_\times\lambda + \phi)}{1 - a^2\cos^2 k_+\lambda - b^2\cos^2(k_\times\lambda + \phi)} \tag{6.3}$$

$$\dot{y} = \frac{1}{c_z}\frac{c_y + c_y a\cos k_+\lambda - c_x b\cos(k_\times\lambda + \phi)}{1 - a^2\cos^2 k_+\lambda - b^2\cos^2(k_\times\lambda + \phi)} \tag{6.4}$$

$$\dot{z} = \frac{1}{2c_z}(c_x\dot{x} + c_y\dot{y}) - \frac{c_z^2 - 1}{2c_z^2} \tag{6.5}$$

$$\dot{t} = \frac{1}{2c_z}(c_x\dot{x} + c_y\dot{y}) + \frac{c_z^2 + 1}{2c_z^2} \tag{6.6}$$

We need c_x, c_y and c_z to determine the motion.

7 Other approach

Light can be treated as photon and EM wave as well. An EM field carries energy, according to Einstein equation, anything carries energy can distort the spacetime. Therefore, when GW and EMW collides with each other, EMW would alter the spacetime, in other words, the spacetime is not the original GW. Meanwhile, spacetime tells the EM field how to propagate. In a simpler terminology, this is the EMW-curvature coupling. Suppose the space is filled with electromagnetic field only, so the stress-energy tensor can be expressed by the electromagnetic stress-energy tensor

$$T^{\mu\nu} = \frac{1}{\mu_0}[F^{\mu\alpha}F^\nu_\alpha - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}] \tag{7.1}$$

In addition to Einstein equation, Maxwell's equation and perturbation theory, the solutions are given by (Cooperstock, 1968)

$${}_{(1)}F^k_{i,k} = h^{a,b}_i {}_{(0)}F_{ab} + h^{bk}{}_{(0)}F_{ib,k} \tag{7.2}$$

$$ {}_{(1)}F_{ik,m} + {}_{(1)}F_{mi,k} + {}_{(1)}F_{km,i} = 0 \tag{7.3} $$

,where h_{ab} is the perturbation metric, in our case, it is the GW. See (Cooperstock, 1968) for derivation. The zeroth order terms are given by

$$ h_{11} = -h_{33} = h\cos(k_g y - w_g t) \tag{7.4} $$

, which presents a y-propagating GW.

$$ {}_{(0)}F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & 0 \\ E_x & 0 & B_z & 0 \\ E_y & -B_z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.5} $$

, where ${}_{(0)}E_x = {}_{(0)}E\cos\theta\cos((k_x x + k_y y) - wt)$, ${}_{(0)}E_y = {}_{(0)}E\sin\theta\cos((k_x x + k_y y) - wt)$ and ${}_{(0)}B_z = {}_{(0)}E\cos((k_x x + k_y y) - wt)$. Eq.(56) represents EM wave travels on xy-plane. This setup describes collision of GW and EMW. After substituting to Eq.(53) and (54) and manipulate it, one gets

$$ \begin{aligned} {}_{(1)}F_{12,00} - {}_{(1)}F_{12,11} - {}_{(1)}F_{12,22} = & \frac{{}_{(0)}Eh}{2}(k_g^2(\cos\theta + 1) - \frac{1}{2}k_x^2 - \frac{1}{2}k_g k_y)\cos(\vec{k^+}\cdot\vec{r} - \Omega^+ t) \\ & + \frac{{}_{(0)}Eh}{2}(k_g^2(\cos\theta + 1) - \frac{1}{2}k_x^2 + \frac{1}{2}k_g k_y)\cos(\vec{k^-}\cdot\vec{r} - \Omega^- t) \end{aligned} \tag{7.6} $$

, where $\vec{k^+} = (k_x, k_y + k_g, 0)$, $\Omega^+ = w + w_g$, $\vec{k^-} = (k_x, k_y - k_g, 0)$ and $\Omega^- = w - w_g$. Eq.(7.6) means the original EMW generates two more waves due to the presence of GW. Their propagation directions are represented by $\vec{k^+}$ and $\vec{k^-}$ .The generated waves are weak as the magnitude of GW h is small.

7.1 Discussion

Do the two other waves indeed exist or it is just flaw generated by perturbation theory? Indeed, (Montanari, 1998) criticized such method is incorrect. In that paper, Montanari indicates that perturbation theory changed some intrinsic property of the problem, or the crucial mathematical structure of the original equation, which leads to non-physical result. Although I worked out the result, I have no confidence about any implication of it. I included this part for anyone interested.

8 Future work

Geodesics are the motion of photons when the spacetime is well given, in other words, the presence of photons does not contribute any curvature of spacetime. This may not be always correct although the effect is always too tiny to be considered. For instance, a spinning object like pulsars is known to be off the geodesics and the motion is governed by Mathisson-Papapetrou-Dixon (MPD) equations (Mathisson, 1937; Papapetrou, 1951; Dixon, 1970). Photon is a spin-half particle, Does the spin make it off the geodesic? Can MPD equations apply on massless particles like photons? These are the questions we want to answer in the future.

Besides photons, light can also be seen as EM waves. This gives us another view or method to deal with the problem of light-GW interaction. Like geodesics of photon, which means non-coupling of photon and spacetime, we can have a test EM field (the concept of test particle) to propagate in this given spacetime. Furthermore, coupling of EM field and spacetime is more obvious as the energy density of EM field can be easily written down and substituted into Einstein equation to illustrate the coupling. To conclude, herein I suggested 4 methods: treat light as photons, and we have non-coupling and coupling case; treat light as EM waves, and we have non-coupling and coupling case as well.

It is interesting that one single problem can be treated by so many different ways. Do they all lead to a single result? We have not known yet, and his project is still ongoing.

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