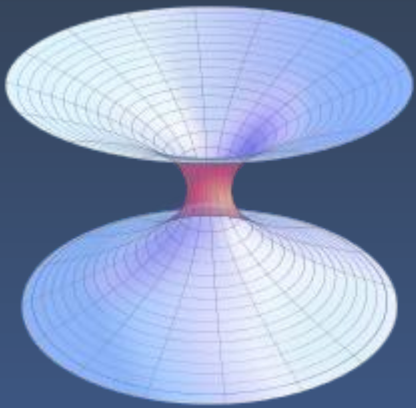
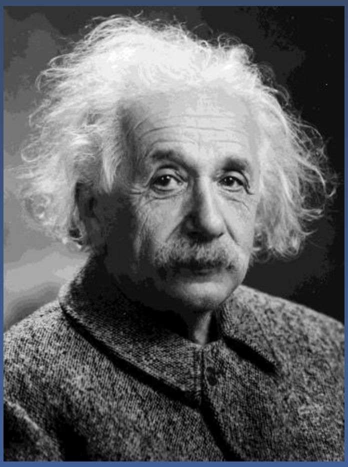
For Physics / Combined Science

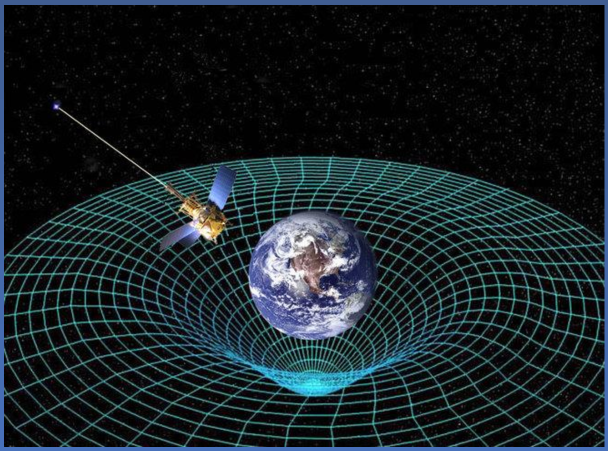
NSS Physics Insight

A short introduction to Special Relativity

Enrichment Topics for HKDSE







KY Li . YY Pang

Consultant : PK Leung, LM Lin, MC Chu

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CUHK Physics

NSS Physics Insight – A short introduction to Special Relativity

NSS Physics Insight – A short introduction to Special Relativity is a textbook for HKDSE physics or combined science students. The topic included in this textbook are mainly about Special relativity, with several supplementary mathematics topics to enforce students with the necessary mathematics tools to deal with SR problems.

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Authors : Li Ka-Yue Alvin, Pang Yiu-Yung
Consultant : PK Leung, Lin Lap Ming, MC Chu

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NSS Physics Insight

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Preface

Li Ka Yue Alvin, Pang Yiu Yung

NSS Physics Insight – A short introduction to Special Relativity is a self-study textbook. It is designed for NSS students and junior undergraduate students.

As a Physics undergraduate student, I have never imagined publishing an e-book in my 4-years of studies. Here, I must deliver my thanks to those who helped me with this project.

This e-book was originally the project work for Dr Lin Lap Ming's course (PHYS3420 – Topics in Contemporary Physics). For Physics courses (except for experimental courses), it is really rare that students have to do a project. Dr Lin has truly provided us an opportunity to exercise our special relativity knowledge, as well as creativity. I would like to thank Dr Lin here, for without without Dr Lin's course, this e-book would not have even appeared! Dr Lin has also encouraged us to present our work at the 1st CUHK Physics Student Conference, which made our work become known to many physics students and teachers at CUHK, and finally lead to this golden opportunity of publishing this e-book.

After the conference, Professor Chu Ming Chung and Dr Leung Po Kin explained that our work is somehow similar in nature to what they are doing for a e-learning project, and granted us the chance to further polish our work and publish our work together in their project. Since last year's June, we have been working with the e-book and Dr Leung and Professor Chu have given us a lot of useful advice and opinions throughout this journey. I would like to express my gratitude of thanks to them.

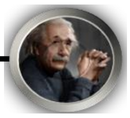
My partner Yiu Yung is another person I must show my appreciation to. Without his help and support, I may not be able to complete this whole work. I must also thank him for his creativity and persistence.

As for myself, I have learnt a lot throughout this project, including how to type equations quickly in Microsoft Office, how to make illustrations using Keynote in Mac, and of course knowledge about Special Relativity. Many of my Physics teachers have always said that being a teacher is the best way to learn Physics, and now I understand why. Before you teach, you must make sure what you write and teach is correct. If you want your readers / students to understand what you are teaching, you must first make it clear to yourself first. Here is a quote from ***Albert Einstein*** *that I really like :*

"If you can't explain it simply, you don't understand it enough…"

Long story short. I better end my preface here before I try to write more. Last but not least, allow me to thank everyone who helped with this project, or simply those who changed my life once again.

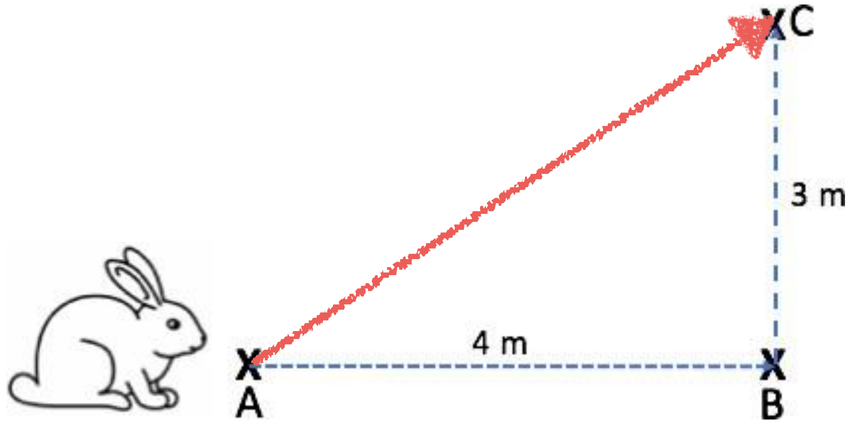
Chapter 1 Pre-Requisite Knowledge



1.1 - Review on motions

In this section we will review some of the important concepts and physical quantities used to describe motions.

Let's consider a smart rabbit below.



The rabbit first moves 4 m from point **A** to point **B**, then moves 3 m from point **B** to point **C**. What is the **distance travelled** by the rabbit?

Distance travelled is the total length of path taken by the observed object. It is a **scalar** quantity which consists of **magnitude**.

In this case, the distance travelled by the rabbit is just : $\underline{4\text{m} + 3\text{m} = 7\text{m}}$

In physics, we often care about the **displacement** of the observed object more. **Displacement** is the distance between the starting position and final position of the observed object. It is a **vector** quantity which consists of both **magnitude** and **direction**.

In this case,

Starting point of the rabbit : Point A   Ending Point of the rabbit : Point C

The **displacement vector** is the red arrow in the diagram above.

The magnitude of the displacement **s** is given by the **Pythagoras Theorem**, which is

$$s = \sqrt{3^2 + 4^2} = 5 \text{ m}$$

And the direction of the displacement is given by $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

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Let's suppose the rabbit takes 2 s to run from point **A** to point **C** via point **B**. What is the **average speed** of the rabbit?

Speed is a scalar quantity which shows how fast the observed object moves along its path of motion. Generally, it is the distance travelled by the object divided by the travelling time.

$$Speed = \frac{Distance\ Travelled}{Travelling\ Time}$$

In this case, the speed of the rabbit is just : $\frac{7\text{m}}{2\text{s}} = 3.5 \text{ m s}^{-1}$

In Physics, we often care more about the **velocity** of the object than its **speed**.

Velocity is a vector quantity which shows how fast is the change in displacement of the object. It consist of both magnitude and direction.

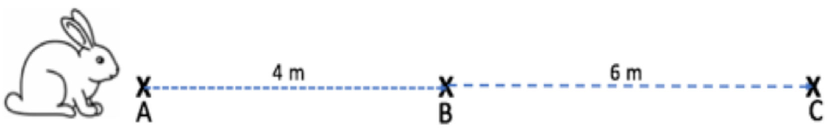
Generally speaking, the direction of the **velocity vector** is the same as that of the **displacement vector**. The magnitude of the velocity vector is just :

$$\text{Velocity} = \frac{Displacement}{Travelling\ Time}$$

In our case, the average velocity of the rabbit is just : $\frac{5\text{m}}{2\text{s}} = 2.5 \text{ m s}^{-1}$

Example 1.1

This time, the rabbit walks along a straight line ABC. It moves from A to B for 2 s, and from B to C for another 2 s.



(a) Find the **total distance travelled** of the rabbit from A to C.
(b) Find the **total displacement** of the rabbit from A to C.
(c) Find the **velocity** of the rabbit along AB.
(d) Find the **velocity** of the rabbit along BC.
(e) Is your answer in (c) and (d) the same? If not, find the difference of the 2 values.

[Solutions]
(a) Total distance travelled = 4 + 6 = 10 m
(b) Total displacement = 10 m (to the right)
(c) Velocity along AB = 4 / 2 = 2 m s^{-1} (to the right)
(d) Velocity along AB = 6 / 2 = 3 m s^{-1} (to the right)
(e) No. The difference is + 1 m s^{-1}

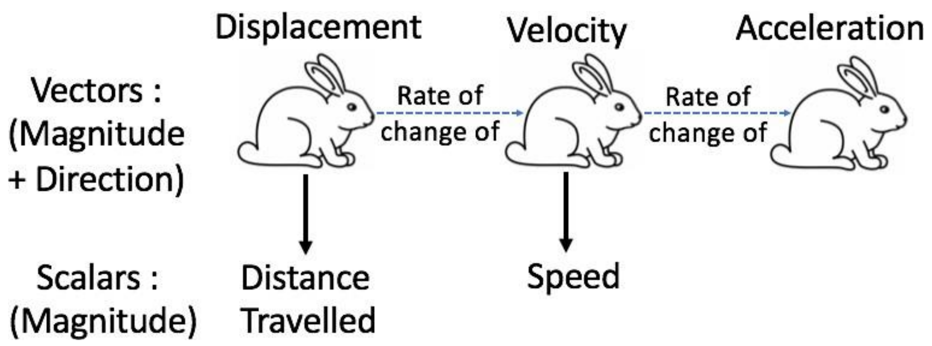
From Example 1.1, we see that the rabbit's velocity increases along AC. The rate of change of velocity of an object is called **acceleration**. We define **acceleration** as

$$Acceleration = \frac{Change\ in\ Velocity}{Time\ difference}$$

For instance, if the rabbit in Example 1.1 changes its velocity from 2 m / s to 3 m / s in 0.5 s time, then the acceleration of it will be $\frac{3-2}{0.5} = 2\ \mathrm{m\ s^{-2}}$

Acceleration is a vector quantity. This follows that even if the object is moving at constant speed, if it is changing direction, then its acceleration is also **non-zero**.

The following schematic diagram summarises the relations between **distance travelled**, **displacement**, **speed**, **velocity** and **acceleration**.



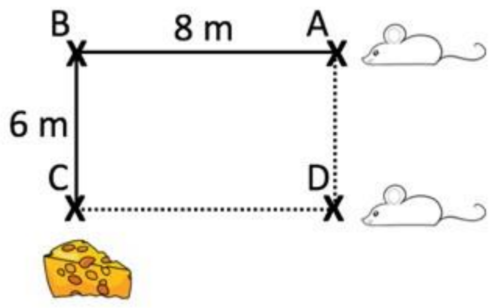
Challenge 1.1

1. Which of the following are not vectors?
   A. Displacement
   B. Acceleration
   C. Rate of change of displacement
   D. Speed

2. Peter walks leftwards 50 m, and then 60 m rightwards. What is his net displacement? (Take right as positive)
   A. 110 m B. 50 m
   C. + 10 m D. – 10 m

3. In a 100 m hurdles competition, Yiu Yung finishes the race in 8 s. Find the average speed of Yiu Yung.
   A. 800 m / s B. 12.5 m / s
   C. – 12.5 m / s D. – 800 m / s

4. Refer to the following diagram. A mouse at A is walking towards a piece of cheese through a path passing through B and C.



(a) Find the displacement of the mouse along the whole journey.
(b) Draw the displacement vector on the diagram above.
(c) If the mouse uses 10 s to reach the cheese, find the velocity of the mouse.
(d) Suppose at t = 0 (The mouse at A starts to move), another mouse at D also walks towards the cheese along DC. Find the **minimum speed** of the mouse at D such that it can reach the cheese before the mouse at A.

If the motion of the observed object is under **uniform acceleration** (i.e. the acceleration is constant), then we have 4 equations of motion describing the object.

We will just state the equations here without proof. You are, however, encouraged to proof the equations of motion for uniformly accelerated motion in the **Problem Set** of this Chapter.

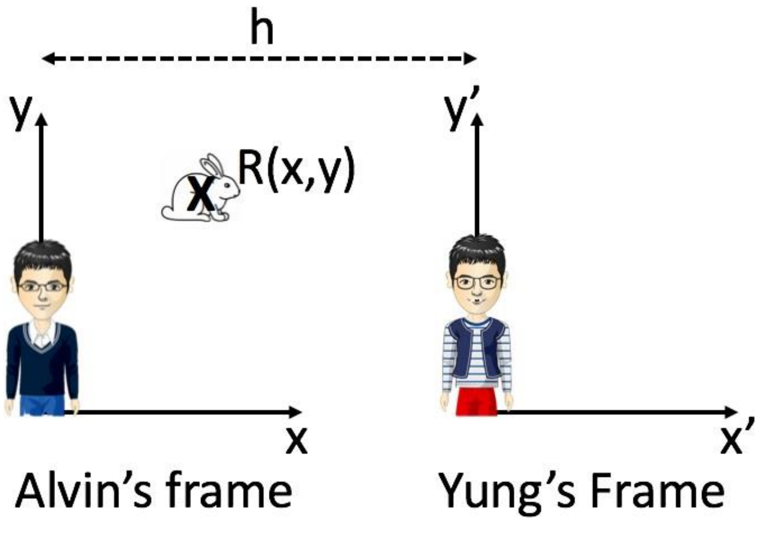
$v = u + at$	$s = ut + \frac{1}{2}at^2$
$a = \frac{v - u}{t}$	$v^2 - u^2 = 2as$

where u = initial velocity, v = final velocity, t = time elapsed, a = acceleration, s = displacement

In this section we will illustrate the transformation of displacement and velocity from one inertial frame to another in the **non-relativistic** point of view.

1.2 - Galilean Transform

Let's consider the diagram below.

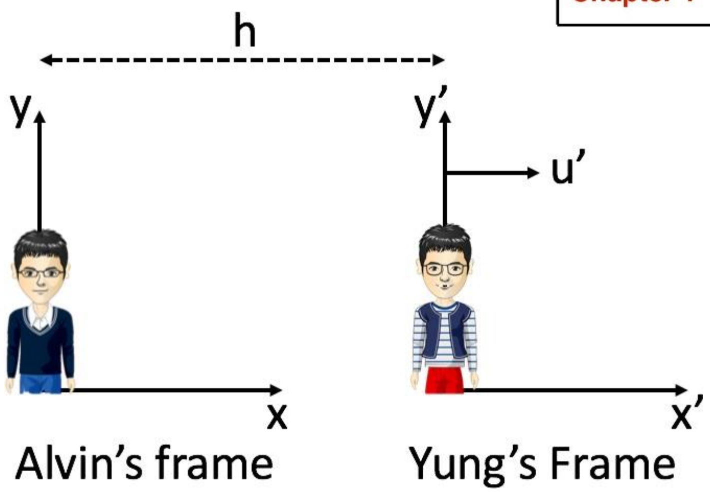


There is a rabbit in Alvin's frame. According to him, the rabbit has position R (x, y).

Alvin's friend, Yiu Yung, is at a distance rightwards from Alvin. What would be the coordinates of the rabbit in Yung's Frame?

It is obvious that for the y' coordinate, it is same as that of y.
But for the x'-coordinate, it will be changed to $x' = -h + x$.
$x' =$ (Try to sketch on the diagram above to make yourself clear!)

Thus the coordinates of the rabbit in Yung's frame will be R' = (x', y') = (- h + x, y).

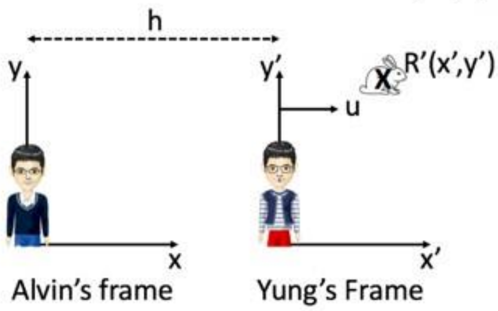


Now suppose Yung is moving to the right at a uniform speed of u' starting from time t = 0.

From Alvin's point of view :	- Initial position of Yung = (x, y) = (________,_______). - After time t, the position of Yung = (x, y) = (____________,_____)
From Yung's point of view :	- Initial position of Alvin = (x', y') = (_________,_______). - After time t, the position of Alvin = (x', y') = (___________,_____)

Example 1.2

This time, let's further assume that there is a rabbit in Yung's frame, which has a coordinate of R' = (x', y').



(a) Find the coordinates of the rabbit R = R(x, y) in Alvin's frame at time t.

(b) Given that $\frac{d(A+B)}{dt} = \frac{dA}{dt} + \frac{dB}{dt}$, and $\frac{d(ct)}{dt} = c$, where c is a constant. Show that the velocity of the rabbit R in Alvin's Frame is given by :

$$v_x = \frac{dx}{dt} = u + \frac{dx'}{dt}$$

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Example 1.2

[Solutions]

(a) R = R (x, y) = (x' + ut, y)

(b) The velocity of the rabbit R in Alvin's frame

$$= v_x = \frac{dx}{dt} = \frac{d(x'+ut)}{dt} = \frac{dx'}{dt} + \frac{d(ut)}{dt} = u + \frac{dx'}{dt}$$

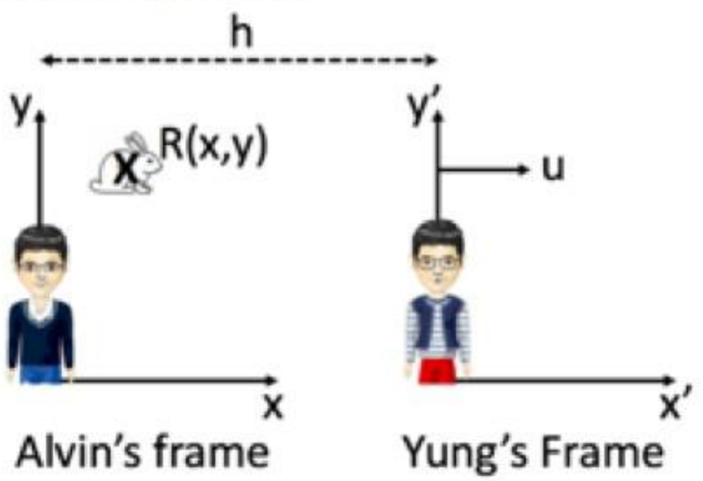
The above equation which relates x and x'are called the **Galilean Transformation** of coordinates.

We will later see that this transformation is useful and good approximation only for speed of objects much smaller than the speed of light c.

At that time, we will need another kind of transformation rule which is called the **Lorentz Transformation**.

Challenge 1.2

1. Refer to the following diagram. Yung is moving to the right at a speed of u. A rabbit is observed in Alvin's frame with position R



(a) Find the coordinates of the rabbit R' in Yung's frame at time t.

(b) Show that the velocity of the rabbit in Yung's frame is given by :

$$v_x' = \frac{dx'}{dt} = -u + \frac{dx}{dt}$$

c) Compare your result in (b) with the result in (b) of Example 1.2. Explain why there is a negative sign in front of the "u" term in (b).

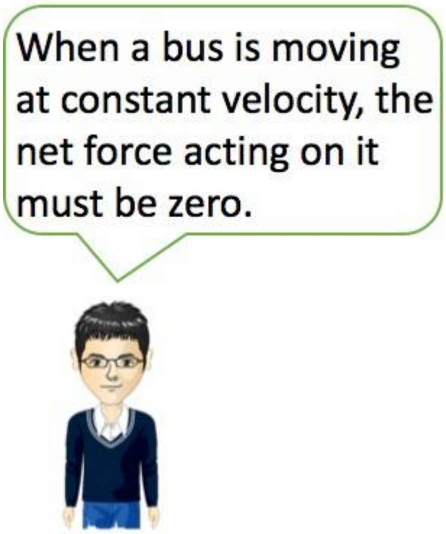
d) Will Yung know that he is moving? Or will he see that Alvin and the rabbit is moving?

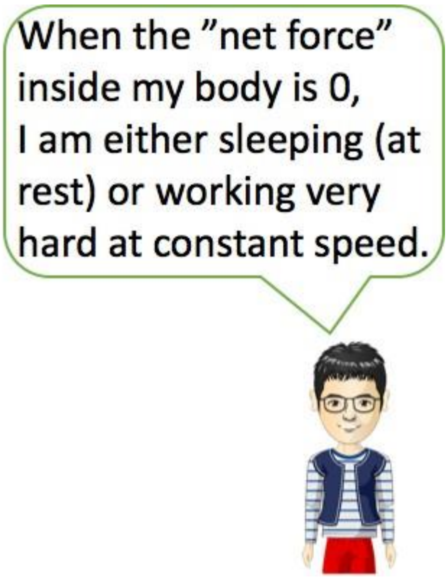
1.3 - Newton's Laws of Motion

In this section we will review about the 3 important laws of motion stated by Newton. They are rather important as in classical mechanics, but it is as important in understand General Relativity.

The 1st Law :

An object is either **at rest** or **in uniform motion** if the **net force** acting on it is **zero**.

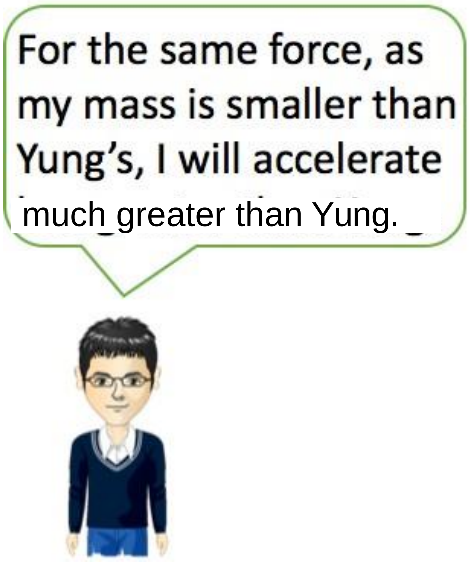


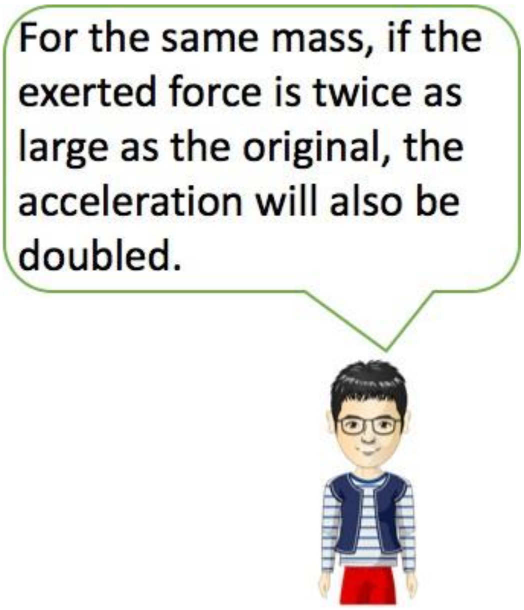


The 2nd Law :

The **acceleration** of an object is **directly proportional** to the **net force** acting on it, and **inversely proportional** to its **mass**. Mathematically,

$$\mathbf{F = km}$$





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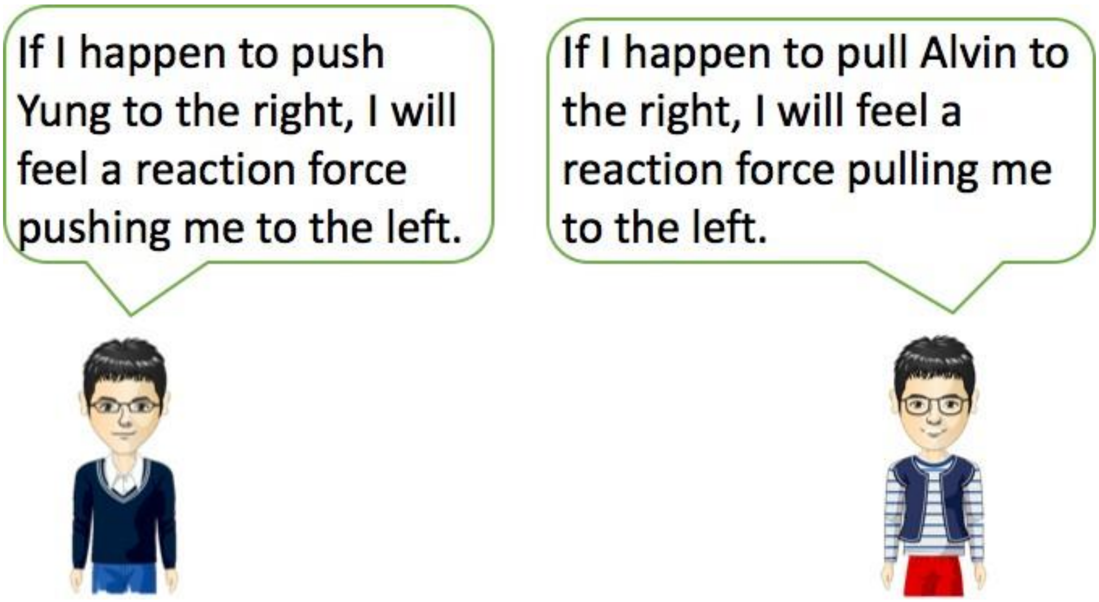
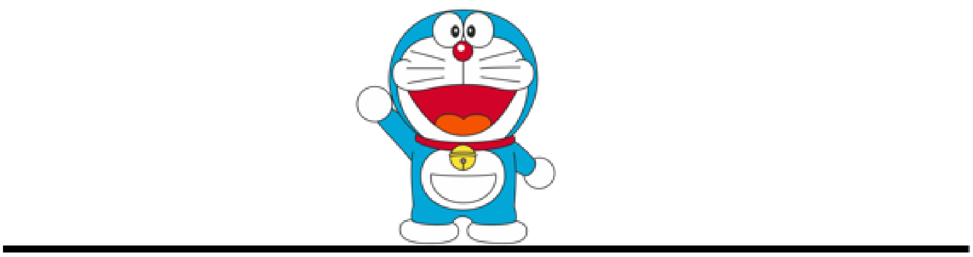
The 3rd Law :

For every **action force**, there must be a **reaction force**. The pair of forces are **equal in magnitude**, **opposite in direction**, **acting on different bodies** and **are of the same nature**.



Example 1.3

Doraemon is now standing firmly on the ground surface.



(a) Draw a free-body diagram, showing all the force acting on Doraemon.

(b) What is the net force acting on Doraemon at the instant shown?

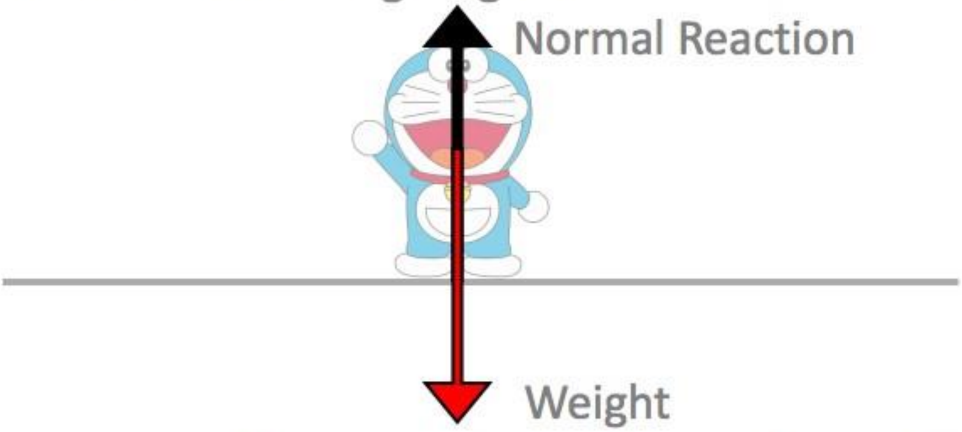
(c) A student claims that the pair of forces in (a) form a pair of action-reaction pair. Comment on his statement.

(d) In fact, Doraemon is standing on the surface of the Earth. Explain whether Doraemon is really "at rest" or not.

Example 1.3

[Solutions]

(a) Refer to the following diagram



(b) According to Newton's 1st Law, the net force is 0.

(c) It is incorrect. Both the normal reaction and the weight act on the same body – Doraemon.

(d) In fact, the Earth is continuously self-rotating and orbiting around the Sun, so in general Doraemon is always changing direction, and hence not at rest.

Challenge 1.3

1. Fill in the blanks :
   (i) An object is either at rest or moving with constant __(a)__ if the net force acting on it is 0.

   (ii) In Newton's 2nd Law, the net force acting on an object is __(b)__ proportional to the mass and acceleration.

   (iii) For every action force, there must be a __(c)__ force which has the same __(d)__, opposite in direction, acting on different bodies and are of the same __(e)__.

2. An astronaut is in a spaceship that is orbiting around the Earth. He claims the net force acting on him is 0 as he is in a state of weightlessness. Comment on his statement.

3. Refer to the following diagram. Batman is climbing a rope which is hung firmly from the ceiling.

(a) Draw a free-body diagram, showing all the forces acting on Batman.

(b) It is given that Batman is instantaneously at rest at the given moment. What is the net force acting on him?

(c) State a pair of forces which have the same magnitude.

(d) If the rope suddenly breaks, what would be the net force acting on batman? Is it zero? Explain your answer.

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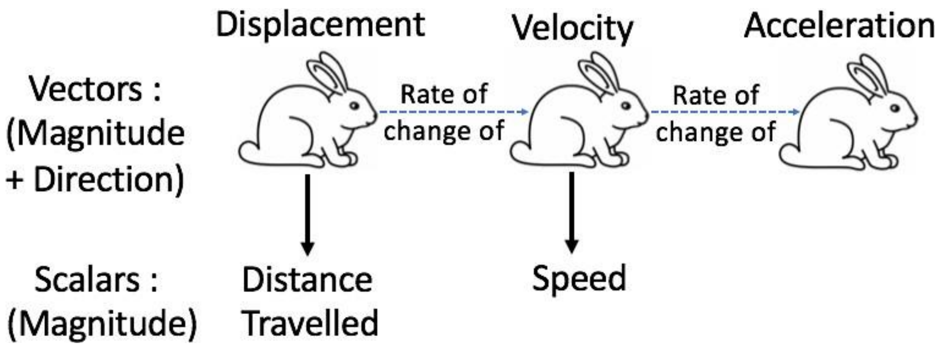
Summary

Key Points

1.1 Review on motion

(a) Distance, displacement, speed, velocity, acceleration



(b) Equations of uniformly accelerated motions

$v = u + at$	$s = ut + \frac{1}{2}at^2$
$a = \frac{v - u}{t}$	$v^2 - u^2 = 2as$

1.3 Newton's Laws of Motion

- An object is either **at rest** or **in uniform motion** if the **net force** acting on it is **zero**.
- The **acceleration** of an object is **directly proportional** to the **net force** acting on it, and **inversely proportional** to its **mass**. Mathematically,

$$\mathbf{F = km}$$

- For every **action force**, there must be a **reaction force**. The pair of forces are **equal in magnitude**, **opposite in direction**, **acting on different bodies** and **are of the same nature**.

Key Terms

100 m hurdles 100 米跨欄	P.5	Acceleration 加速度	P.5
Action force 作用力	P.10	At rest 靜止	P.9
Average 平均	P.4	Displacement 位移	P.3
Galilean Transformation 伽利略變換	P.8	Lorentz Transformation 勞侖茲變換	P.8
Magnitude 數值	P.3	Mass 質量	P.9
Nature 性質	P.9	Net Force 淨力	P.9
Newton 牛頓	P.9	Opposite 相反	P.10
Proportional 成比例	P.9	Reaction Force 反作用力	P.10
Speed 速度	P.4	Uniform acceleration 勻加速度	P.6
Vector 矢量	P.3	Velocity 速率	P.4

Chapter Exercise

Multiple Choice Questions

1. A person standing on the ground. Which following statements are correct?
   1. The gravitational force acting on you by the Earth and the force acting on you by the ground are the action and reaction pairs.
   2. The gravitational force acting on you by the Earth is equal to the reaction force acting on you by the ground.
   3. There is no reaction force acting on you when you are punching other.

   A. 1 only
   B. 2 only
   C. 1 and 2
   D. 2 and 3

2. The bus travelling on the road with an acceleration. What is the change in velocity of the bus after 5 seconds for 5ms^{-2} and 2 seconds for -5ms^{-2} ?

   A. 15ms^{-1}
   B. 25ms^{-1}
   C. -10ms^{-1}
   D. 35ms^{-1}

3. John and Kelly are running to same direction from the park. John starts 2 minutes earlier than Kelly and his speed is 18km/h. It is known that Kelly's speed is twice that John. When would Kelly can pass John?

   A. 1 minutes
   B. 2 minutes
   C. 3 minutes
   D. 4 minutes

4. From Galileo's experiment, which following statement is correct?

   A. The speed is proportional to the object's weight.
   B. The speed of the metal object is smaller than other one.
   C. The speed of the object at same height to the ground is the same.
   D. All of above are correct.

Short Questions

1. Under Galilean Transformation, when we are travelling in high speed and try to measure the

speed of light. What values are we can measure?

a) The speed is $500ms^{-1}$.

b) The speed is $10000ms^{-1}$

c) The speed is 0.5c

d) The speed is 1c

(which c is the speed of light)

2. The spaceship is moving at $500ms^{-1}$ toward the Earth. At same time, a missile has been

launching from the Earth toward the spaceship. The collision of them is after 3 minutes later.

(Assume both of them are not affected by external force)

(a) What is the speed of the missile?

(b) Meanwhile, the owner of the spaceship immediately know the missile is coming. He turn the

spaceship opposite direct and try to escape. If the speed of spaceship still the same, will the

missile attack it? If yes, how long after it launched?

Structured Questions

[Question 1]

In a recent TV programme "逃げるは恥だが役に立つ", the ending dance was surprisingly popular and attracted many people to learn the dance.

In one part of the dance, the dancer moves 8 steps forward while shaking their hands and fingers. And then jump 9 steps backward.

(a) What is the total distance travelled of the dancer at the end?

(b) What is the total displacement of the dancer at the end?

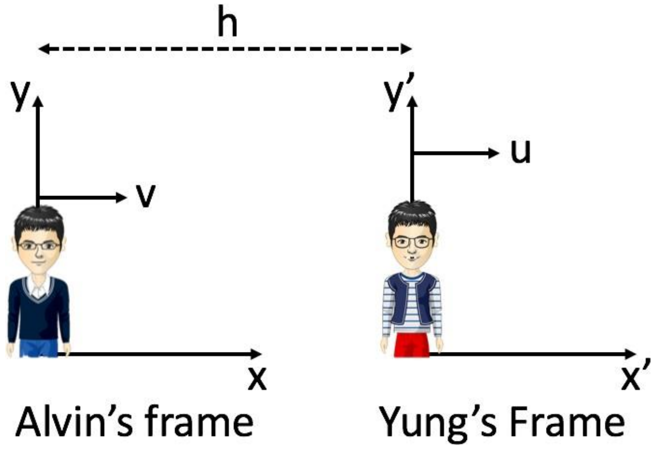
(c) Let's assume that the dancer uses 8 seconds to move forward, and uses 6 seconds to jump 9 steps backward. Find the respective average velocity of the dancer for the forward and backward motion.

(d) Has the dancer accelerated during the forward and backward motion? Explain your answer.

(e) A student claims that if the dancer has an extra step, he / she may have zero displacement. Do you agree? If yes, state whether the extra step should be in the forward motion or in the backward motion.

[Question 2]

Consider the following diagram. Yung is at a distance h rightwards from Alvin. Yung is moving to the right at a speed of u, while Alvin is moving right at a speed of v.



(a) Let's consider the case of v = u. From Alvin's point of view, will Yung move away from him, move towards him or will he be at rest?

(b) Let's consider the case of v > u. Assume initially Yung's coordinate in Alvin's frame is $Y = Y\,(h, y)$. Find Yung's coordinates after time t in Alvin's Frame. From Alvin's point of view, will Yung move away from him, move towards him or will he be at rest?

(c) Let's consider the case of v < u. Assume initially Yung's coordinate in Alvin's frame is $Y = Y\,(h, y)$. Find Yung's coordinates after time t in Alvin's Frame. From Alvin's point of view, will Yung move away from him, move towards him or will he be at rest?

(d) Redo (a) - (c) from the point of view of Yung. Assume that initially Alvin's coordinates in Yung's frame is $A = A\,(-h\,, y')$

[Question 3]

According to myths, the famous scientist Galileo once conducted an experiment on the tower of Pisa by throwing 2 objects of different masses onto the ground. He wanted to show that the acceleration of objects is independent of their masses.

(a) Draw a free body diagram for one of the object, showing all the forces acting on it. You may neglect air resistance.

(b) State Newton's 1st Law of Motion, and hence explain whether the object is at rest or in uniform motion.

(c) A student claims that in this case, there is no action-reaction force pair concerning the falling object. Comment on his statement.

(d) If Galileo used a metal ball and a very light feather for his experiment, what would be the result? Does it violates Galileo's conclusion? Explain your answer.

[Question 4]

The man try to across the river which has width 50 meter. The speed of the water flow is flowing to downward.

(a) The man swims at 2m/s

(i) How long does he arrive the opposite river bank?

(ii) What is his vertical displacement?

(iii) What is the direction of his final position?

(b) The man swims to upward and has an angle 40^{o} to the river flow.

(i) If he has no vertical displacement when he arrive the opposite side, what is his speed?

(ii) If he swims to downward and keep the angle, what is his vertical displacement?

[Question 5]

Flash man is running on the one direction road. A snipper, who is behind 150 meters from Flash man, is trying to shoot Flash man. It is given that the speed of bullet is 1200 m/s.

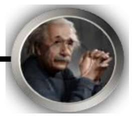
(a) What is the time for the bullet to hit Flash man who travels in 100 m/s?

(b) If Flash man starts to speed up before the bullet just hit, what is the speed of Flash man at least accelerated to?

(c) If Flash man keep moving at 200 m/s, his girlfriend(standing in front of him 50 meters) try to say something to him. Does he hear his girlfriend's voice first or hit by the bullet first? Take the speed of sound is 340 m/s.

THE END

Chapter 2 Relativistic Time and Length



In this section we will introduce the 2 important postulates in Einstein's theory of Special Relativity. These lay the foundation for the development of the theory.

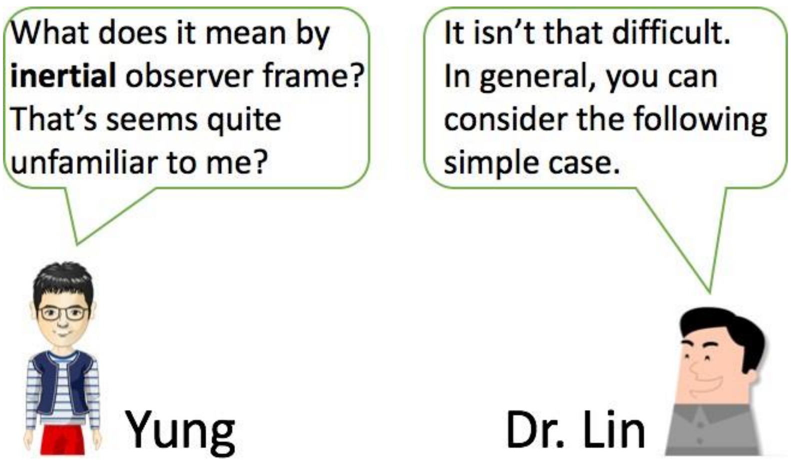
2.1 - Postulates of Einstein's Special Relativity Theory

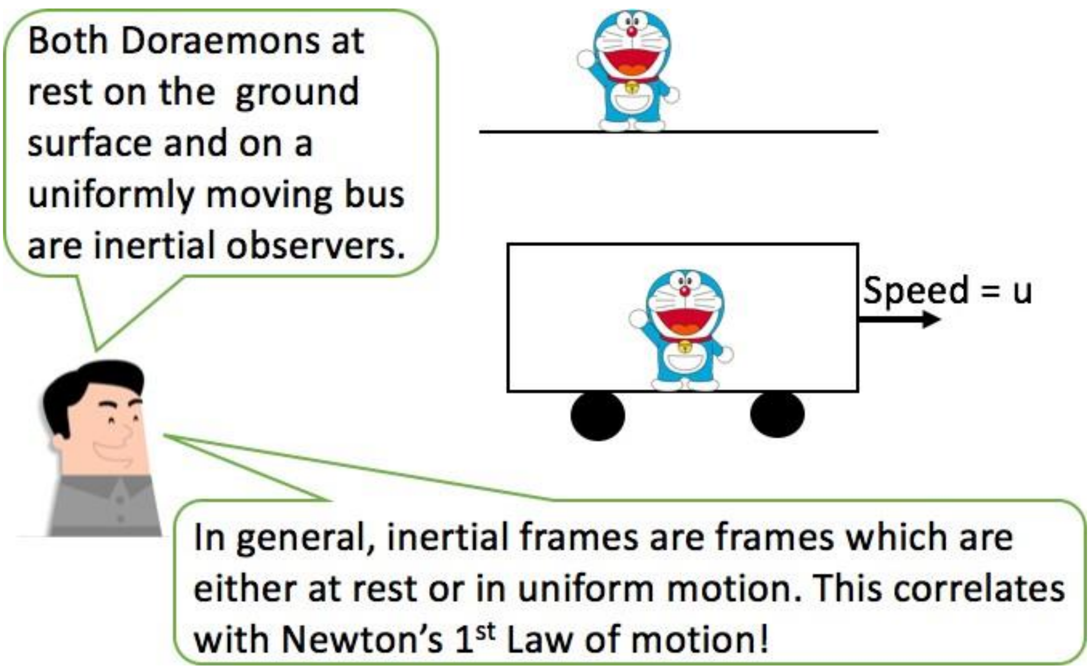
A **postulate** is an assumption made for a theory or a law. The theory / law will be correct only if the postulates are accepted to be true.

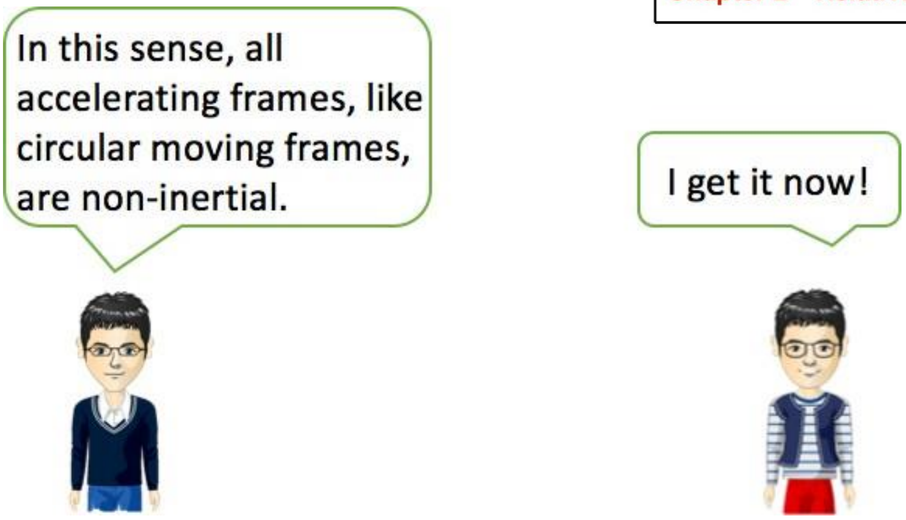
In Einstein's Special Theory of Relativity, he laid down 2 important postulates. Let us now go through the postulates one by one.

<u>The 1st Postulate :</u>

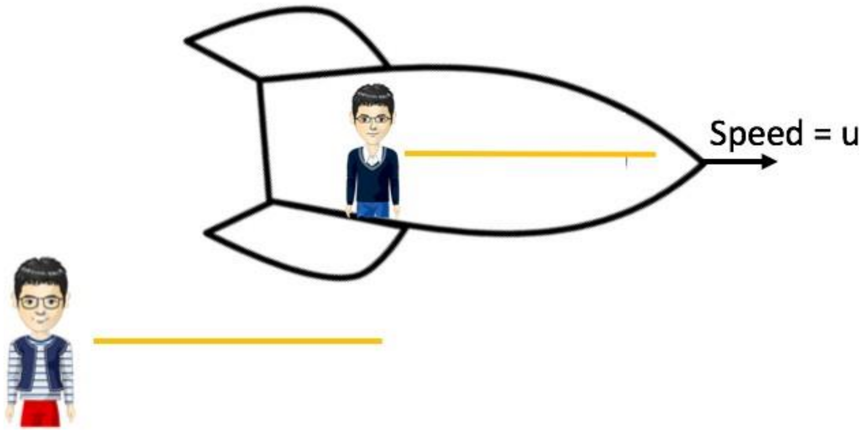
The speed of light is a constant (c) in all **inertial** observer frames.







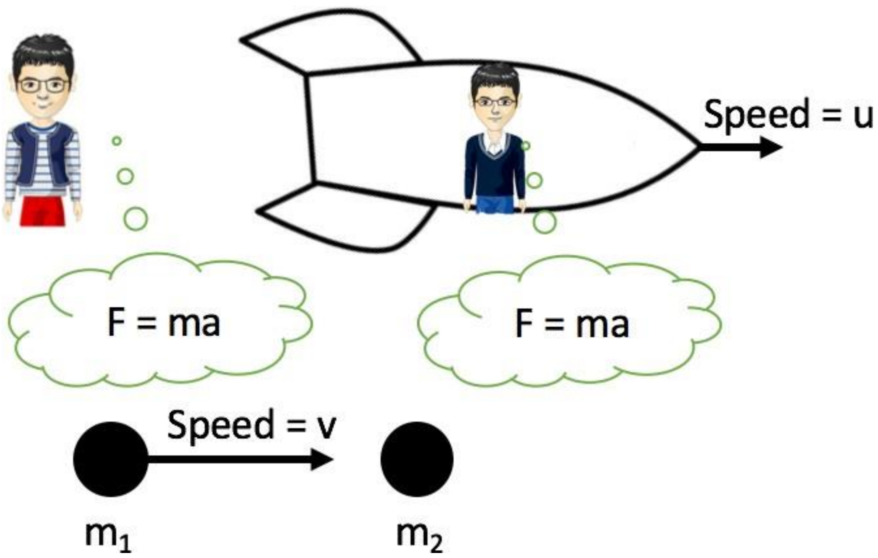
It is important to note that even circular frames, which moves in uniform speed, are **non-inertial**, because it is always changing its direction of motion, and it is thus accelerating!



Using Einstein's postulate, the speed of light (yellow beam) is constant at $c = 3 \times 10^{8}\ \mathrm{m\ s^{-1}}$ in both inertial frames.

<u>The 2nd Postulate :</u>

The laws of physics are **invariant** in all inertial observer frames.



Einstein also postulated that the physics we have been dealing with, like the usual F = ma, electrostatics, waves and other theories are all the same in all inertial frame.

Although the mathematics might be slightly different, the overall result should remain the same. The all-accepted laws, like the conservation law of energy and momentum, should not change when we switch from a rest frame to a uniformly moving frame.

Challenge 2.1

1. Which of the following(s) are inertial observer?
   (a) A man sitting on a chair.
   (b) A driver driving a bus at uniform velocity.
   (c) A child cycling in circular motion around a centre.
   (d) A bird free-falling from the top of a building.
2. State whether the speed of light is equal to the constant "c" or is undetermined in each of the following cases :
   (a) A boy swimming along a straight line with uniform speed sees a beam of light passing.
   (b) An astronaut in an accelerating spaceship sees a beam of light reaching his ship.
   (c) A man running at light speed "c" towards a light source sees a ray of light coming towards him.
3. You are now standing on the Earth's surface. Answer the following questions using this assumption.

   a) State the law of conservation of energy.
   b) State Einstein's postulates for his Special Relativity theory.
   c) According to your answer in (b), would an astronaut in a spaceship moving in uniform velocity sees the same physics as stated in (a)? How about an astronaut in another ship which accelerates at constant acceleration?
   d) What is the condition for an inertial frame? By considering the Earth's daily and yearly motion, explain whether you are an inertial observer or not.
   e) Would it lead to any contradiction to our daily observation if your answer in (d) is "**no**"? Try to think about it.

In the last question of Challenge 2.1, you are asked about whether or not you are an inertial observer.

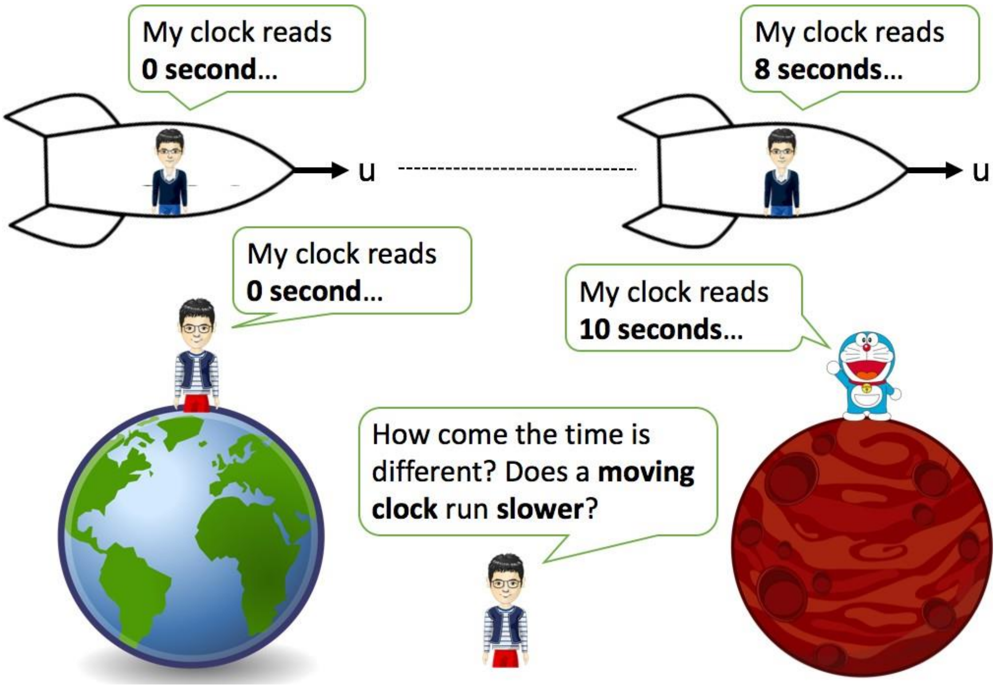
Certainly, if you consider the daily and yearly motion of the Earth, as it is always rotating, it is obvious that we are always accelerating, and thus we are not inertial observers.

However, if we consider a small enough part of our Earth, such as the laboratory which we conduct all the physics experiment, it can be regarded as a locally inertial frame, so the laws of physics can still be held true, and will not lead to any contradictions.

You will come again to the idea of local inertial frame when you study General Relativity.

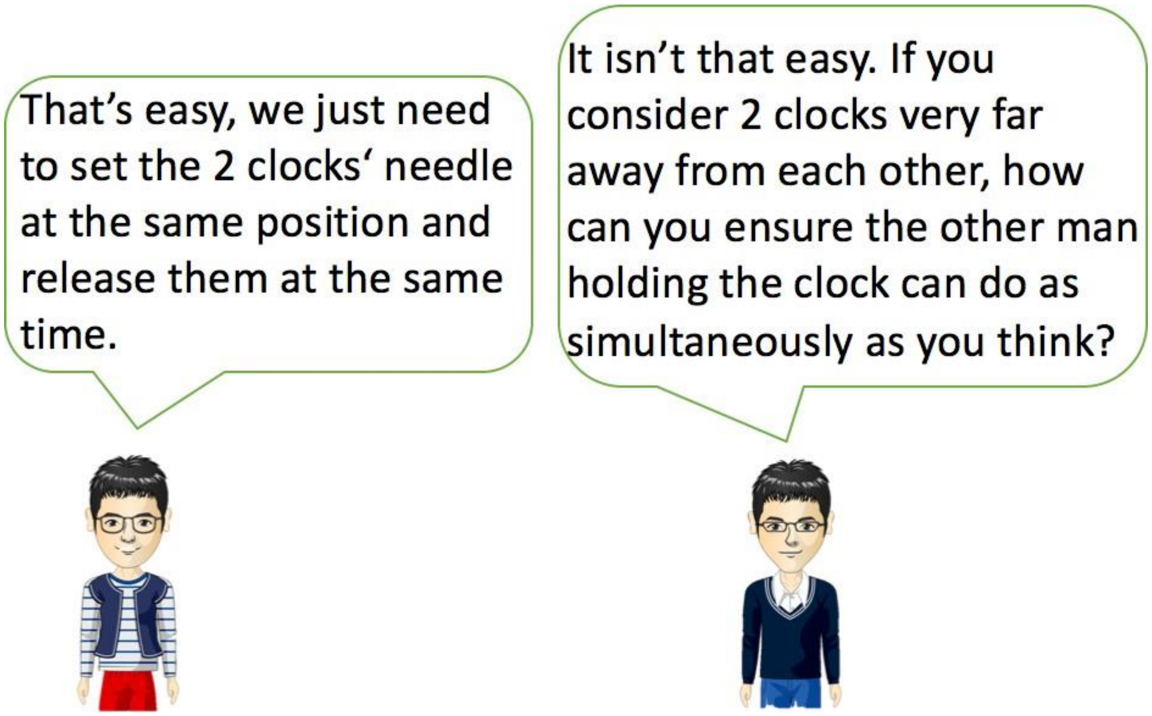
2.2 - Time Dilation

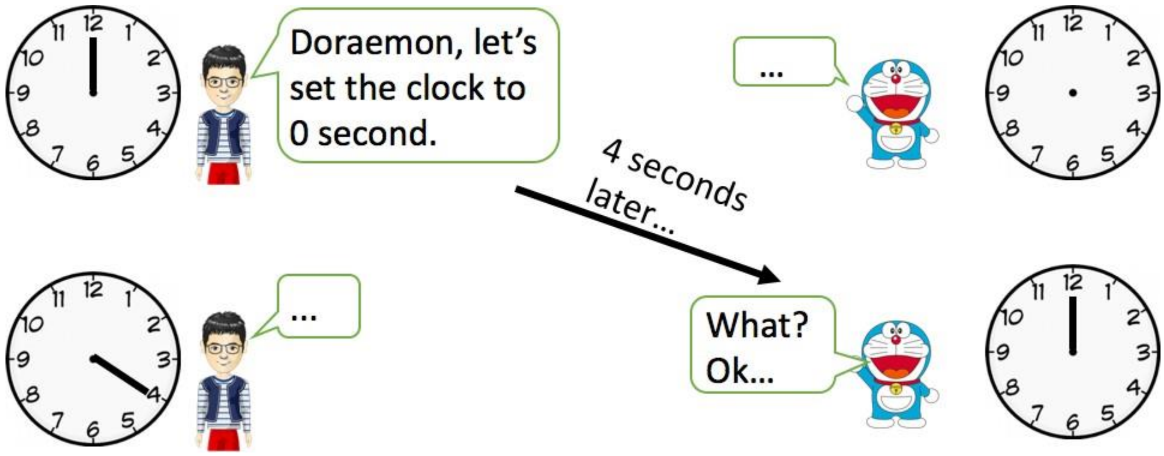
In this section we will show how we can deduce the time dilation equation. We will also illustrate the idea of synchronised clocks and proper time.



What has happened in the above case? To answer this question we first need to understand the concept of synchronised clocks.

What means by synchronisation? It means we need to make the clocks "ticks" at the same time.

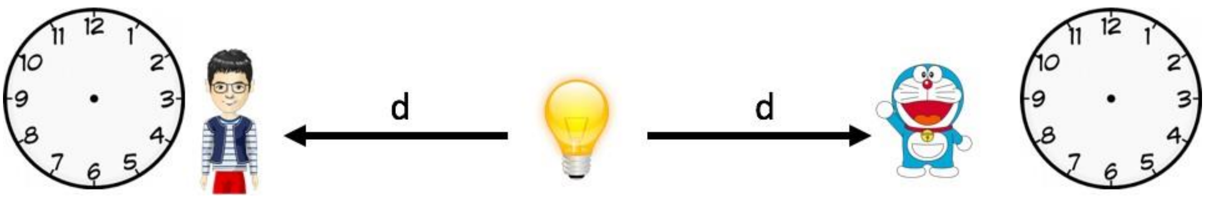




It would be quite unrealistic if you think you can synchronise clock easily, because even light needs time to travel from one place to another, and thus it will lead to time dilation problem when you try to synchronise clocks.

So how actually do one synchronise clocks? We shall discuss one method here.

One way to synchronise clocks is to start them together. Here is how it's done :



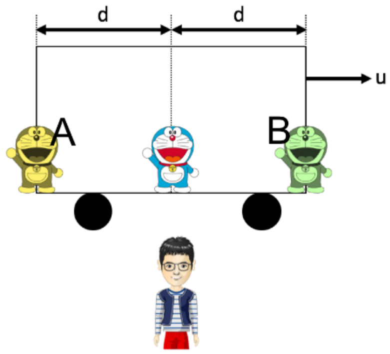
We place a **light source** in the middle of the 2 clocks. We then turn on the light source and it will send 2 light signals to both the clocks.

Once the clocks receive the signals, it will **automatically** start counting time. In this sense, we can ensure that the clocks are synchronised.

It is notable that **simultaneity** is **NOT** an absolute idea in relativity. We shall illustrate this idea in the following example :

Example 2.1

At the instant shown, a Doraemon is standing in the middle of a moving bus of speed u. Two of his friends, A and B are standing at the 2 ends of the bus. Yiu Yung is at rest outside the bus.

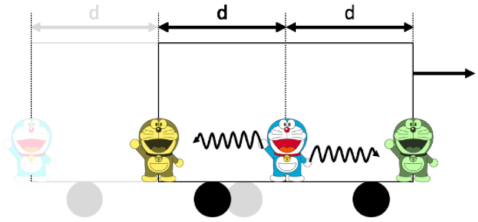


(a) At time t = 0, Doraemon sends 2 light signals to A and B simultaneously. In his point of view, will A and B receive the signals at the same time? Or else who will receive the signal first?

(b) Repeat (a) using Yiu Yung's point of view.

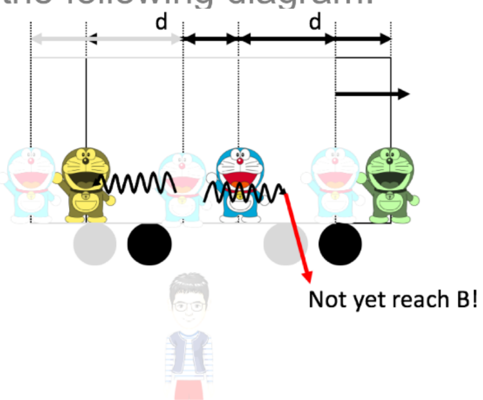
[Solutions]

(a) Consider the following diagram.



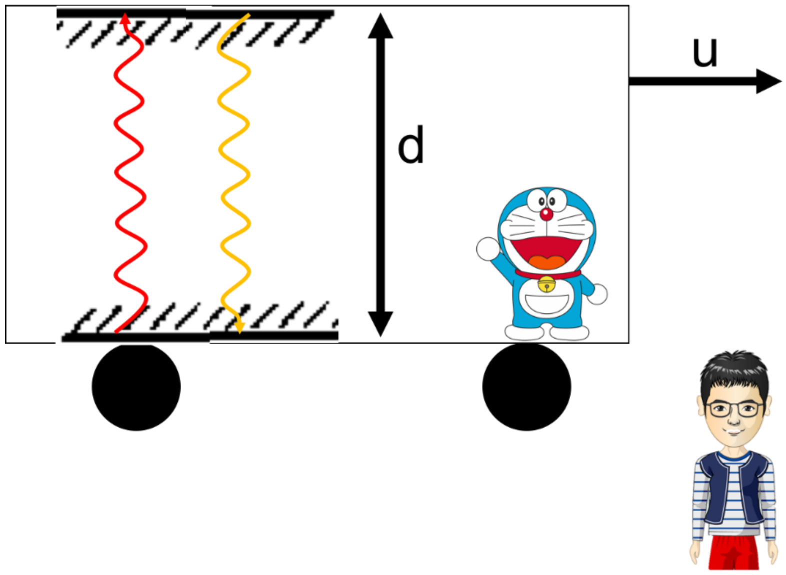
From the figure, we can see that in Doraemon's frame, A and B will receive the signal at the same time.

(b) Consider the following diagram.



From the figure, we can see that in Yiu Yung's frame, A and B will NOT receive the signal at the same time. A will receive the signal earlier than B.

Let's consider the following case :



Inside a moving car, there are 2 mirrors.

At time t = 0, a light signal is sent from the bottom mirror to the upper mirror. After being reflected from the upper mirror, it returns to the bottom mirror, and stops the time counter.

How much time will have elapsed at the end as seen by Doraemon?

[Solutions]

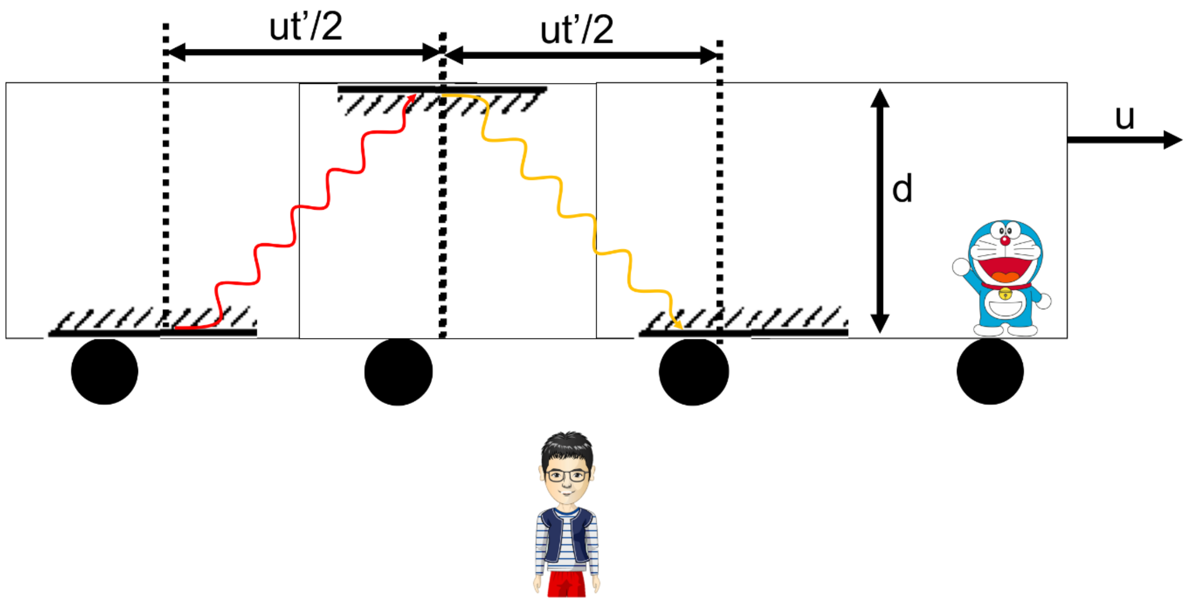
The total distance travelled by the light **s** = ___________

The speed of light in all inertial frame = ________

The time elapsed in Doraemon's frame **t** =___________— (1)

———————————————————————————————————————

As seen from Yiu Yung's frame,



The light signal does not travel along a vertical straight paths, but rather 2 diagonals.

How much time will have elapsed at the end as seen by Yiu Yung?

[Solutions]

The total distance travelled by the light **s**

The speed of light in all inertial frame = _________

The time elapsed in Yiu Yung's frame **t'**

——

If you compare the 2 time intervals, **t** and **t'**, you will see that you can actually connect the both results using one equation :

$$t'^2 = \frac{4d^2}{c^2 - u^2}$$

$$t'^2 = \frac{\frac{4d^2}{c^2}}{1 - \frac{u^2}{c^2}}$$

$$t'^2 = \frac{t^2}{1 - \frac{u^2}{c^2}}$$

$$t' = \frac{t}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma t$$

where µ is called the **Lorentz Factor** which we will come across again in Chapter 3. It is defined as :

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

The equation :

$$t' = \frac{t}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma t$$

is called the "Time dilation" equation.

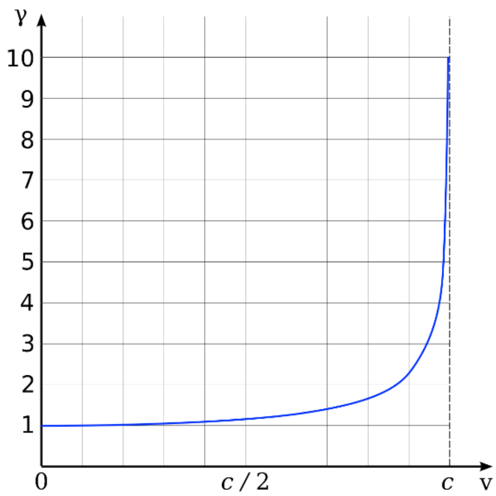
It is now clear why it would seems that "a moving clock" runs slower! Since u is always < c, the Lorentz factor will always be larger than 1.

The time as measured by Doraemon, **t**, will hence always be smaller than the time measured by Yiu Yung, **t'**.

We usually called the time interval between 2 events that occurs at the same point in space as the **proper time**.

We may also refer the proper time as the time interval measured by the same clock, while the time interval which requires the use of **2 or more clocks** as the **improper time**.

Hence, the proper time in this case is the time interval measured by Doraemon.



It is interesting to note that, when u tends to 0, the Lorentz factor tends to 1, which leads to the result that **t** = **t'**.

Typically, when u << c/2, the Lorentz factor is very close to 1.

We call the range of values u << c as the **non-relativistic** zone in physics. In such regions, relativistic effect is not important and can be neglected.

Relativistic effect is only obvious when u ~ c.

Example 2.2

Bolt (保特) is often refer to the fastest running man in the world. In the 2009 Berlin Olympics, he made the World Record of completing a 100 m race in 9.58 s.

(a) Evaluate the speed of Bolt in the 100 m race.

(b) Compute the Lorentz factor for this case. You may approximate the result by :

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \approx 1 + \frac{u^2}{2c^2}$$

and write your answer as 1 + a, where a is a constant.

(c) Suppose Bolt held a clock with him while he was racing, and he measured time t at the end. Find $\frac{t'}{t}$ where t' = 9.58. Assume t = 0 at the start of the race.

(d) Evaluate the percentage error of the time measured by the time-recorder and Bolt.

[Solutions]

(a) The speed = $\frac{100}{9.58}$ = 10.438 m / s

(b) $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \approx 1 + \frac{u^2}{2c^2} = 1 + \frac{10.438^2}{2c^2} = 1 + 1.21\times10^{-15}$

(c) $t' = \gamma t = (1 + 1.21\times10^{-15})t$

$$\gamma = \frac{t'}{t} = \frac{1}{(1+1.21\times10^{-15})}$$

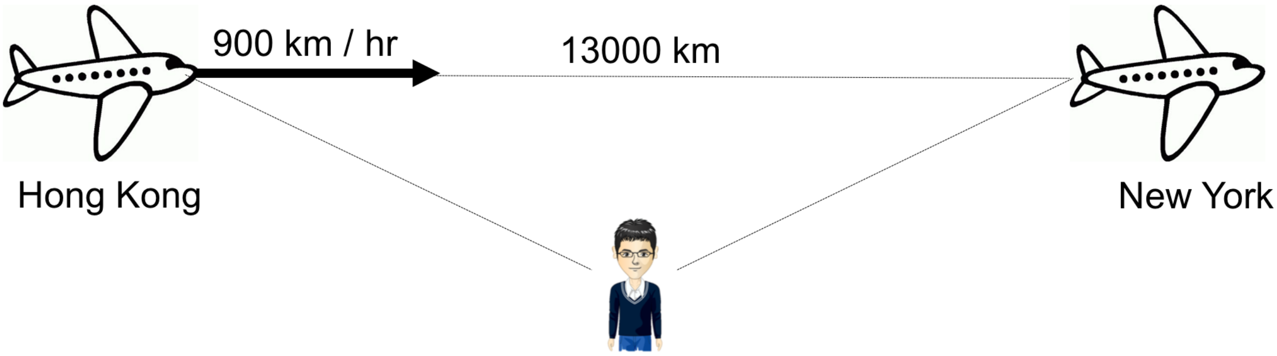
(d) Percentage error

$$\frac{t'-t}{t} = \frac{t'}{t} - 1 = \gamma - 1$$

$$= \frac{1}{(1+1.21\times10^{-15})} - 1 \approx 1.21\times10^{-13}\ \%$$

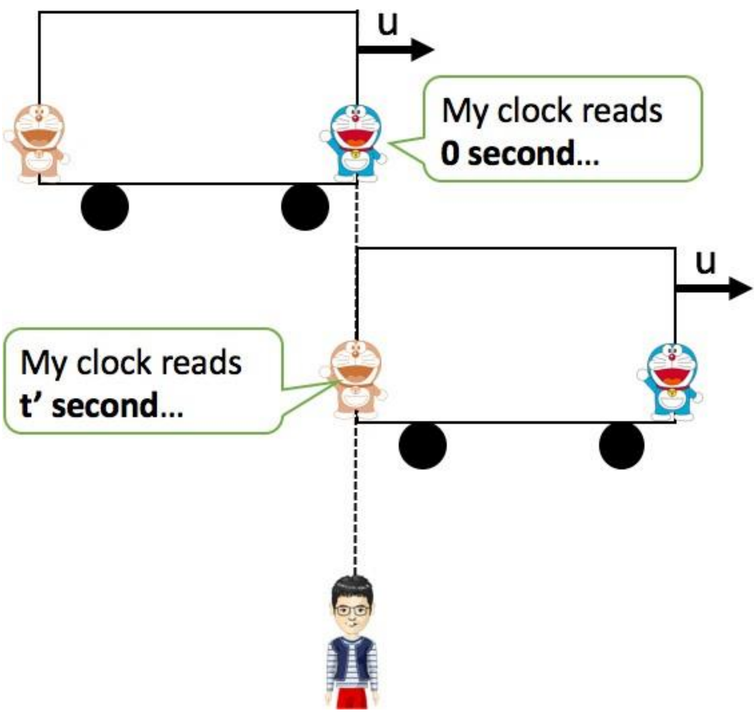
Challenge 2.2

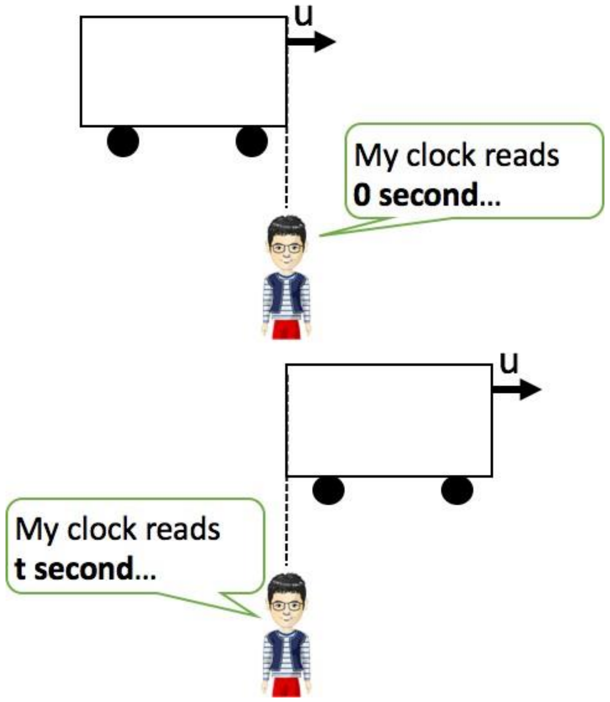
1. A plane is flying from Hong Kong to New York at a speed of 900 km / hr. The distance between Hong Kong and New York is about 13000 km. The pilot of the plane uses a clock to measure the time of flight. Another inertial observer on the ground also measures the time interval.



(a) What is the time of flight as measured by the inertial observer on the ground?
(b) What is the time of flight as measured by the pilot?
(c) Which observer measures the proper time? Explain your answer.

In this section we will show how we can deduce the length contraction equation. We will also illustrate the relations between relativistic time and length.





In the above case,________________(Doraemon / Yiu Yung) measured the proper time.

In relativity, we define the length measured by a person **co-moving** with the object to be measured as the **proper length**.

In the above case, the proper length **Lo** is the length measured by Doraemon, but Doraemon measures the **improper time**.

The length measured by Doraemon is :

$$L_0 = ut'$$

On the other hand, Yiu Yung measures the **proper time** but he measures the **improper length**.

The length measured by Yiu Yung is :

$$L = ut$$

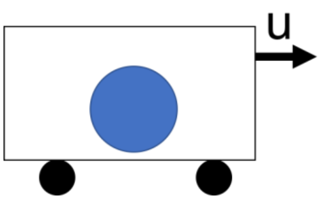
If we combine the above 2 equations, we can get :

$$\frac{L_0}{L} = \frac{ut'}{ut} = \frac{t'}{t} = \gamma$$

$$L = \frac{L_0}{\gamma}$$

Hence, we can see that the proper length of objects is contracted if it is measured by an observer who is not co-moving with it. We call this effect **length contraction**.

It is notable that only lengths which are along the direction of motion will be contracted. For instance :

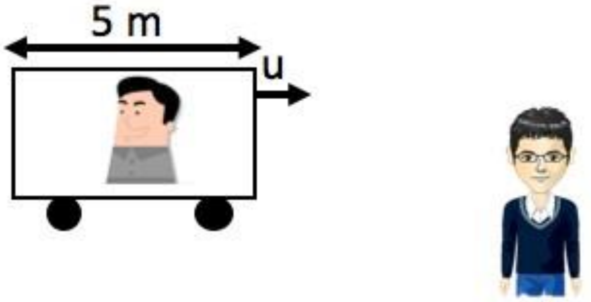


In the eye of an observer on the ground, the object in blue will be seen as :

only the width of the object is contracted, but the height remains the same.

Example 2.3

Let's consider a typical bus of length 5 m. Dr. Lin is sitting on the bus while Alvin is on the ground.



(a) What is the proper length of the bus?

(b) If the bus is moving along a straight line at speed u, what would be the length measured by Alvin? Work out your steps clearly.

(c) How fast should the bus move we want the measured length by Alvin to be half of the original length of the bus? How about one-third of the original length?

[Solutions]

(a) The proper length = 5 m.

(b) The length measured by Alvin is the **improper length** L, so we have $L = \frac{L_0}{\gamma} = 5\sqrt{1 - \frac{u^2}{c^2}}$

(c) If we want the measured length by Alvin to be half of the original length, then we have :

$$\frac{5}{2} = 5\sqrt{1 - \frac{u^2}{c^2}}$$

$$u = \frac{\sqrt{3}}{2}c$$

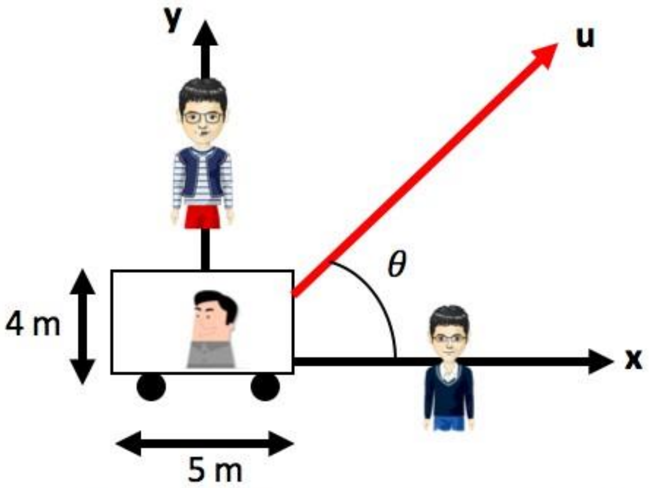
If we want the measured length by Alvin to be 1/3 of the original length, then we have :

$$\frac{5}{3} = 5\sqrt{1 - \frac{u^2}{c^2}}$$

$$u = \frac{2\sqrt{2}}{3}c$$

Challenge 2.3

1. Let's us redo example 2.3 with a 2-D situation.



This time, the bus moves at a speed of u along a path which makes an angle θ with the horizontal x-axis. Alvin is on the x –axis while Yiu Yung is on the y-axis.

a) Write down the expression for the length of the bus measured by Alvin and Yiu Yung respectively, in terms of u and θ.

b) Let us set θ to be 45°. At what speed of u will both Yiu Yung and Alvin measure the lengths of the bus to be three-fourth of the original length.

c) Use the speed u you evaluate in (b). At what angle can the measured length by Yiu Yung to be 4/5 of the original? What will be the measured length of the bus by Alvin at this speed and angle?

d) Consider a new coordinate system u-v with Dr. Lin's bus as the origin. How would Yiu Yung and Alvin move on this new coordinate grid? Sketch a diagram to illustrate your answer.

e) If θ = 90°, what will be observed by Alvin? Explain your answer.

f) Suppose that Yiu Yung is in fact moving towards the positive y-direction. Explain whether he can measure length contraction of Dr. Lin's bus.

Summary

Key Points

2.1 Postulates of Einstein's Special Relativity Theory

(a)Postulates of Special Theory of Relativity

The 1st Postulate :

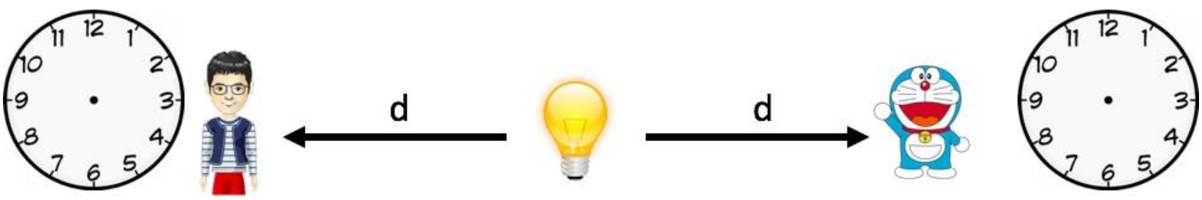
The speed of light is a constant (c) in all **inertial** observer frames.

The 2nd Postulate :

The laws of physics are **invariant** in all inertial observer frames.

2.2 Time Dilation

a) Synchronisation of clocks :



We place a **light source** in the middle of the 2 clocks. We then turn on the light source and it will send 2 light signals to both the clocks.

Once the clocks receive the signals, it will **automatically** start counting time. In this sense, we can ensure that the clocks are synchronised.

(b) Time Dilation Equation :

$$t' = \frac{t}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma t$$

2.3 Length contraction

(a)Length Contraction Equation :

$$L = \frac{L_0}{\gamma}$$

(b) Lorentz Factor :

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Key Terms

Accelerating	加速中	P.2	Automatically	自動地	P.5
General Relativity	廣義相對論	P.3	Co-moving	同時運動	P.12
Contraction	縮短	P.12	Correlate	使...互相有關係	P.1
Dilation	延長	P.4	Ensure	確保	P.4
Inertial	慣性	P.1	Invariant	不相關	P.2
Laboratory	實驗室	P.3	Light source	光源	P.5
Locally inertial frame	局部慣性坐標系統	P.3	Lorentz Factor	勞倫茲因數	P.9
Non-relativistic	非相對	P.10	Postulate	假設	P.1
Proper length	在相對於觀察者而言是靜止的座標系中所量到的長度	P.12	Proper Time	在相對於時鐘是靜止的座標系所量到的時距	P.9
Simultaneously	同時	P.4	Synchronisation	同步	P.4
Tends to	漸趨於	P.9	Unrealistic	不現實	P.5

Chapter Exercise

Multiple Choice Questions

1. If you siting in the rocket travelling close to speed of light. Which following is correct?
   A. Having infinite life time
   B. Being a fat/tall guy
   C. Your time is longer than others by measuring from you than the guy in the Earth.
   D. Your time is shorter than others by measuring from you than the guy in the Earth.

2. Choose the best answer to fulfill the following situation. Peter measured a super-hero across the sky. He thought speed of the hero was 2000 meters per second.
   A. The hero seemed taller.
   B. The hero seemed shorter.
   C. The hero was a point.
   D. Peter can't see the hero.

3. Find the proper length of the rocket when it is moving in c/2 relative to the Earth, where c is the speed of light, the one on Earth measured its length is L/3.
   A. $\frac{\sqrt{3}}{2}L$
   B. $\frac{1}{2}L$
   C. $\frac{2}{\sqrt{3}}L$
   D. L

4. The proper time of the rocket in Q3 is T, what is the time inside the rocket.
   A. $\frac{\sqrt{3}}{2}T$
   B. $\frac{1}{2}T$
   C. $\frac{2}{\sqrt{3}}T$
   D. T

Short Questions

1. Complete the following summary.

(a) Einstein proposed 2 postulates for his Special Theory of Relativity. The 1st one is that, the __i__of light is constant in all inertial reference frames. The 2nd one is that, the laws of physics are___ii__in all inertial reference frames.

(b) In relativity,___iii__is NOT an absolute idea. That is, it is not always true for 2 events to happen at the same time if we consider 2 different inertial reference frames.

(c) The mystery of “a moving clock runs slower” can be solved by the explanation of time____iv___. The time interval between 2 events measured at the same position in space is called the_v___.

(d) It is an important idea to___vi__clocks before use in relativity. We have to make sure the clocks we use are in phase.

(e) Proper length refers to the length of the object which is measured by an observer__vii___with the object. If any other observers which is not moving with the object measures the length of the object, the length will be___viii___than expected. This effect is called____ix___.

(f) By consider the Lorentz factor, if the speed of the object is much smaller than the speed of light,_x__effect is not significant and it will reduce to the normal Newtonian result.

Structured Questions

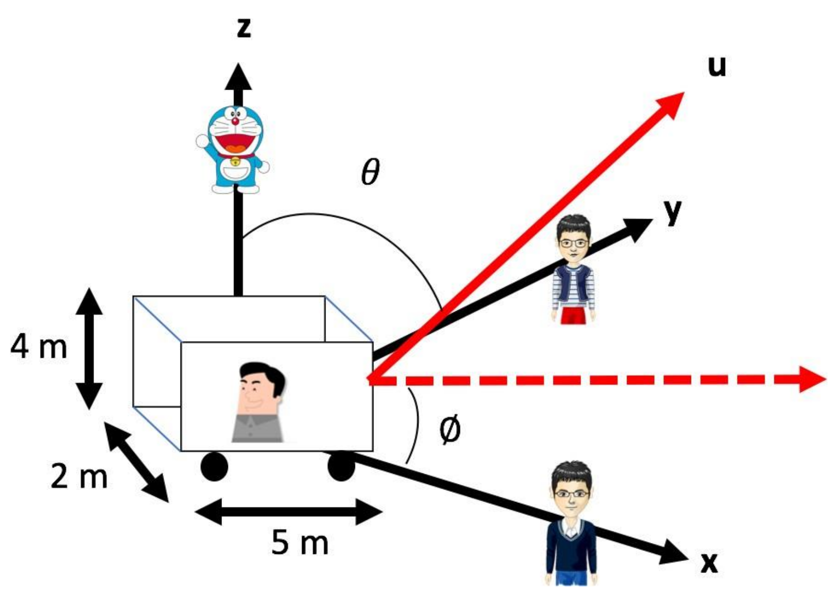
[Question 1]

Try to show your steps clearly in derivate the Lorentz Factor :

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

[Question 2]

Consider the problem in challenge 2.3. We now want to extend our discussion to a 3-D situation.



This time, the bus moves along a straight line in 3D which makes an angle θ with the z - axis and an angle ϕ with the x - axis. Alvin, Yiu Yung and Doraemon stand on the x, y and z - axes respectively.

(a) Express the length of the bus measured by Alvin, Yiu Yung and Doraemon respectively. Label the length as La, Ly and Ld respectively.

(b) Suppose $\theta = 90^o$, what would be the value of Ld? Explain your answer briefly.

(c) Suppose $\phi = 90^o$, what would be the value of La? Explain your answer briefly.

(d) Suppose $\phi = 0^o$, what would be the value of Ly? Explain your answer briefly.

(e) Suppose u = 0.5 c, what should be the size of ϕ and θ if:

(i) We want the length contraction of La to be the largest?

(ii) We want the length contraction of Ly to be the largest?

(iii) We want the length contraction of Ld to be the largest?

(f) Let's suppose u = 0.5 c, $\theta = 45^o$ and $\phi = 45^o$. Define a new quantity

$$(\Delta S)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta t)^2$$

where Δt, Δx, Δy and Δz are the change in time, change in length along x-axis, y-axis and z- axis respectively. Compute the ΔS^2 for Dr. Lin's bus if Δt = 1 s.

[Question 3]

An astronaut in a spaceship travels to Proxima Centauri, which is approximately 4.2 ly away from the Earth. The speed of the spaceship is 0.70 c (c = speed of light in vacuum) relative to the Earth's frame.

(a) Draw a picture to show the situation of the problem.

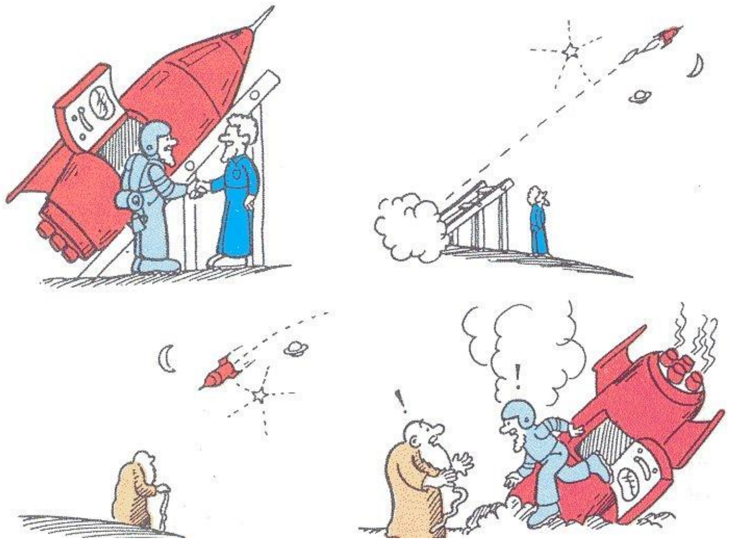
(b) What is the proper length of the distance between Proxima Centauri and the Earth?

(c) How far is Proxima Centauri from the Earth, as seen by the astronaut in the spaceship?

(d) How long does it take for the spaceship to reach Proxima Centauri, as seen by an observer on Earth?

(e) How long does it take for the spaceship to reach Proxima Centauri, as seen by the astronaut in the spaceship?

[Question 4]- [The twins paradox]



There was once a pair of twins Einstein and Newton. One day, Einstein rode on a spaceship and leave Earth at a speed of u (u < c) relative to Earth, and after 5 years later Einstein returns to Earth at the same speed u.

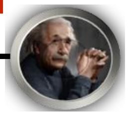
According to Newton, who is on Earth, because "a moving clock runs slower", Newton concluded that Einstein will seem to be younger on his return.

However, in Einstein's frame of reference, his spaceship was at rest while Newton and the Earth is moving away and back to him. Once again, because "a moving clock runs slower", Einstein concluded that Newton will seem to be younger when Einstein return to Earth.

Who is correct? What is wrong with this question? See if you can figure out the flaw in this question. (**Hint** : Think about the postulates of Special Relativity…)

THE END

Chapter 3 Lorentz Transformation



In this section we will show that under the 2 postulates of Special Relativity, the Newtonian Galilean Transformation will lead to problem. This will motivate us to introduce a new transformation rule - The Lorentz Transformation, in the next section,

3.1 - Failure of Galilean Transformation

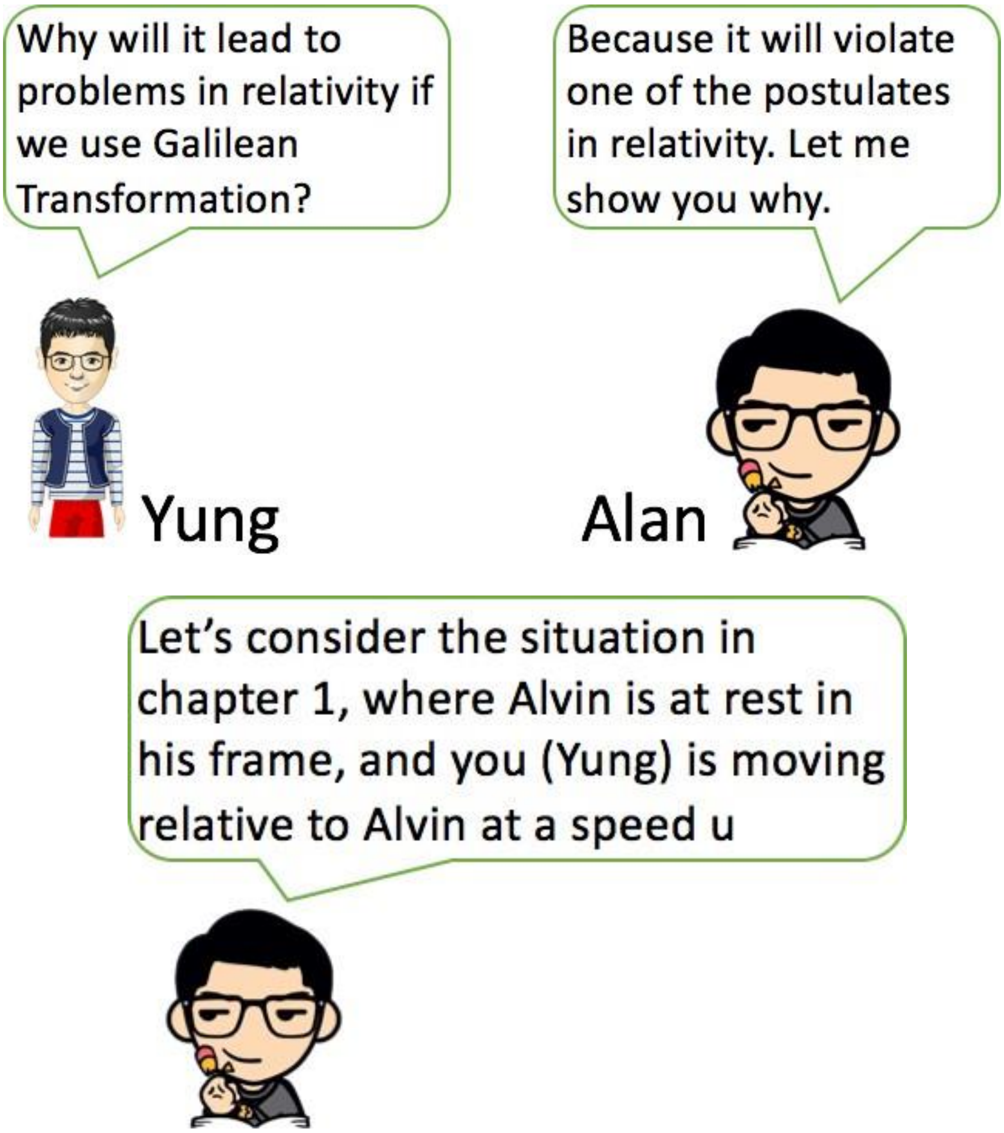
Recall in Chapter 2, we have learnt the 2 important postulates of Einstein's Special Theory,

<u>The 1st Postulate :</u>

The speed of light is a constant (c) in all **inertial** observer frames.

<u>The 2nd Postulate :</u>

The laws of physics are **invariant** in all inertial observer frames.



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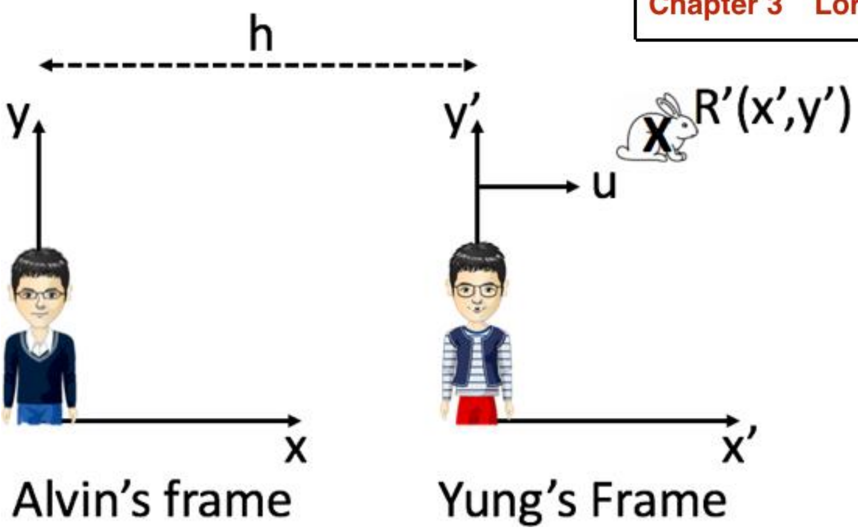

Revision :

Complete the following table :

From Alvin's point of view :	- Initially position of Yung = (x, y) = (_________,_______). - After time t, the position of Yung = (x, y) = (_____________,_____)
From Yung's point of view :	- Initially position of Alvin = (x', y') = (_________,_______). - After time t, the position of Alvin = (x', y') = (_____________,_____)

What will be the position of the rabbit R in Alvin's frame? ______________________________

After some time t, the position of the rabbit R will also change by a small amount x. If we divide both side of the above equation by t, we will get :

$$\frac{\Delta x}{\Delta t} = \frac{u\Delta t}{\Delta t} + \Delta x'\Delta t$$

Recall that in Chapter 1, we have learnt that the rate of change of displacement is known as the **velocity**. Therefore, we can write the above equation as :

$$v_x = u + v_x'$$

where v_x is the velocity of the rabbit in Alvin's frame, u is the velocity of Yung in Alvin's frame, and v_x' is the velocity of the rabbit in Yung's frame.

Now, if the rabbit is moving at a speed of c (speed of light) in Alvin's frame, then,

$$c = u + v_x'$$

and hence

$$v_x' = c - u$$

but it violates Einstein's postulate that, the speed of light should be invariant in all inertial reference frame!

#Note : The speed of light is different in different frame in the above situation!

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That is why the Galilean Transformation rules cannot be used in relativity!

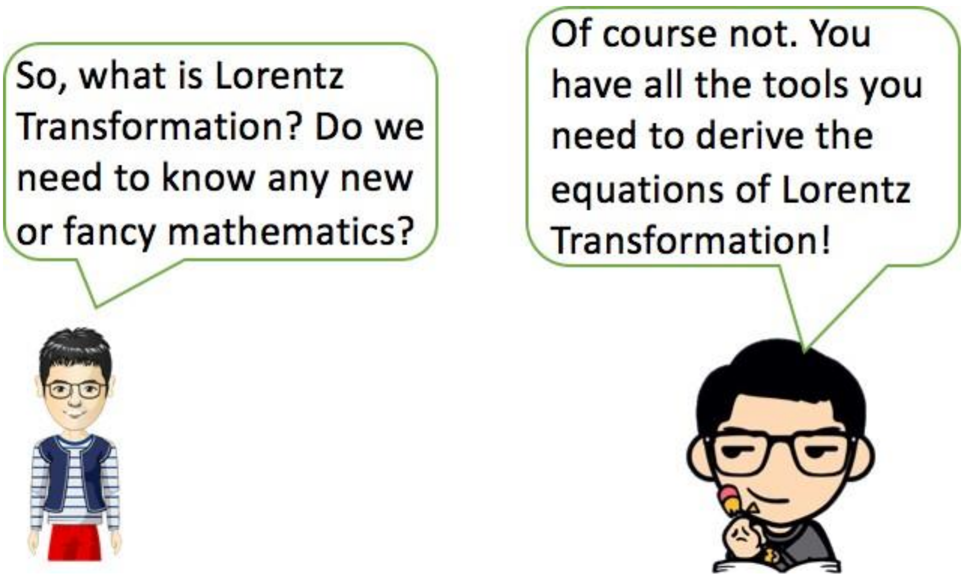
Galilean transformation is only a **good approximation** for speed much lower than the speed of light (non-relativistic), but it will eventually break up when it comes to relativity.

To solve the problem, we will need a need set of transformation rules, which can take into account the relativistic effect, and is known as the **Lorentz Transformation**. You will learn more about it in the next section.

Challenge 3.1

1. The Earth self-rotates about its axis of rotation once every 24 hours. The radius of the Earth is 6370 km. In this question, assume that the Earth is a perfect sphere.

(a) If you stand at rest on a point along the Equator of the Earth, find your speed due to the self-rotation of the Earth and express your answer in m / s.
(b) Suppose there exist a train which can travel at a speed of 0.999999 c (c = speed of light in vacuum) along the equator. Find the speed of the train according to an observer at rest in the outer space using **Galilean Transformation**.
(c) State whether your answer in (b) is physically correct. If not, explain your answer briefly.
(d) A man inside the train in (b) uses an instrument to emit a beam of light opposite to its direction of travel. State the speed of the light beam.
(e) Repeat (d) if the beam of light is emitted along the direction of travel of the man.



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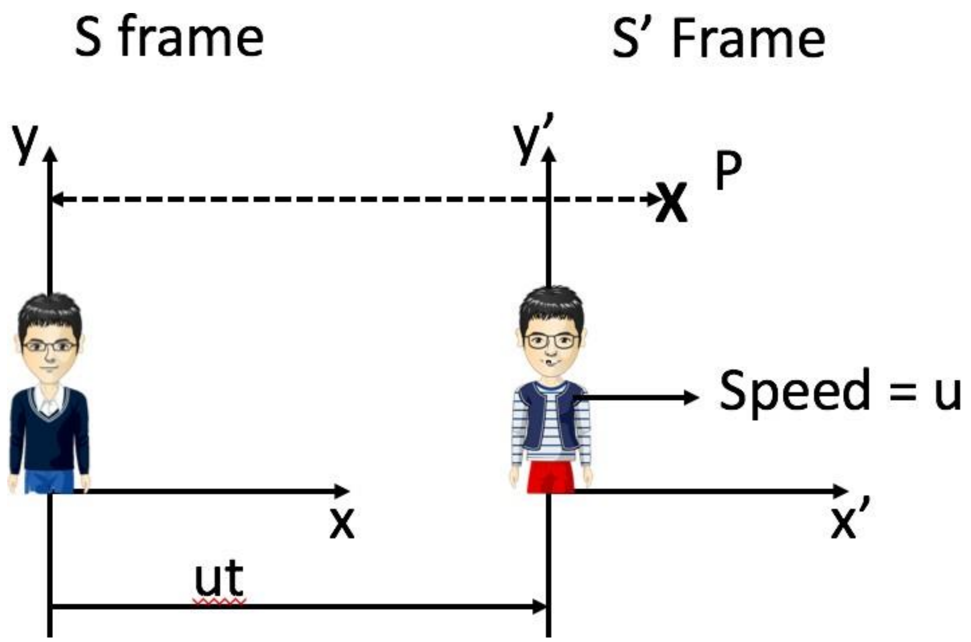
3.2 - Lorentz Length Transformation

In this section we will derive the equations of the Lorentz Transformation of Length.

Let us consider the following situation. The S-frame is at rest, while S'-frame is moving along the positive-x direction at a speed u. There is a point P in the diagram.



In general, there are **4-coordinates** for every point in the **spacetime** : [t , x , y , z]. t refers to the time-coordinate, while x, y and z refer to the spatial coordinates.

Let's call the coordinates of point P in the S-frame as P = (t, x, y, z), and that in the S'-frame as P' = (t', x', y', z').

Note that, x' is the **proper length** of an imaginary horizontal rod joining the y' axis and point P in the S'-frame.

The corresponding **improper length** of that "rod" in the S-frame would be________________.

Hence, x can be related to x' by :

$$x = ut + \frac{x'}{\gamma}$$

where   is the Lorentz factor.

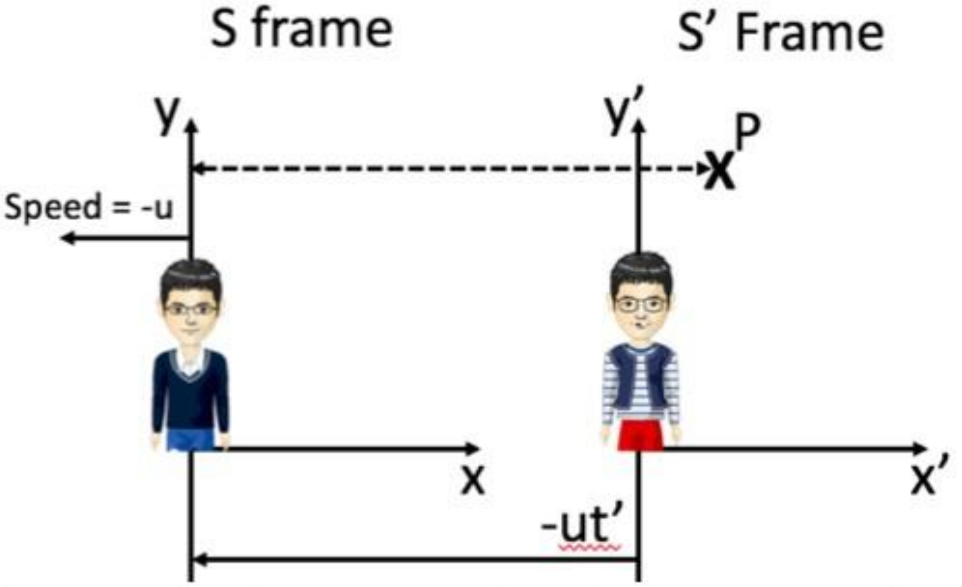
By rearranging the terms, we have :

$$x' = \gamma(x - ut)$$

This is the **first result** of the Lorentz Length transformation.

Example 3.1

Now, we try to redo the above procedures from the point of view of the S' frame observer.



(a) Find a relation between x' and x in terms of ut'.

(b) Make x as the subject of the equation in (a).

[Solutions]

(a) Note that x is the **proper length** of an imaginary horizontal rod joining the y-axis and the point P in the S-frame. The **improper length** in the S' frame would be $\frac{x}{\gamma}$. Hence, the relation can be written as $x' = -ut' + \frac{x}{\gamma}$

(b) $x = \gamma(x' + ut')$

Therefore, we get the 2 important length transformation rules under the Lorentz's transformation.

$$x' = \gamma(x - ut) \quad \text{and} \quad x = \gamma(x' + ut')$$

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Challenge 3.2

1. Recall, from the Lorentz Length Transformation, we have :

$$x' = \gamma(x - ut) \quad \text{and} \quad x = \gamma(x' + ut')$$

(a) Make x as the subject for the equation $x' = \gamma(x - ut)$.
(b) By comparing your result in (a) with the equation x= $\gamma(x' + ut')$, determine an equation relating t and t'.
(Note : The resulting equation is the Lorentz Time Transformation equation.)

(c) Write out explicitly the equation of the Lorentz Factor γ.
(d) Find the value of γ when the speed "u" is much smaller than the speed of light. Hence show that the Lorentz Transformation of Length reduces to the usual Galilean Transformation.

3.3 - Lorentz Time Transformation

In this section we will derive the equations of the Time Transformation under Lorentz Transformation.

As we can see from Challenge 3.2, we have the relation between t and t' (in S-frame and S' frame) as the following :

$$t = \gamma\left(t' + \frac{ux'}{c^2}\right)$$

Similarly, if we make x' as the subject for the equation $x = \gamma(x' + ut')$, and compare it with the equation with $x' = \gamma(x - ut)$, we can get another time-tranformation equation :

$$t' = \gamma\left(t - \frac{ux}{c^2}\right)$$

The above 2 equations are the **time-transformation equations** under the **Lorentz Transformation rules**.

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Summary

Key Points

3.1 Failure of Galilean Transformation

(a) Failure of Galilean Transformation

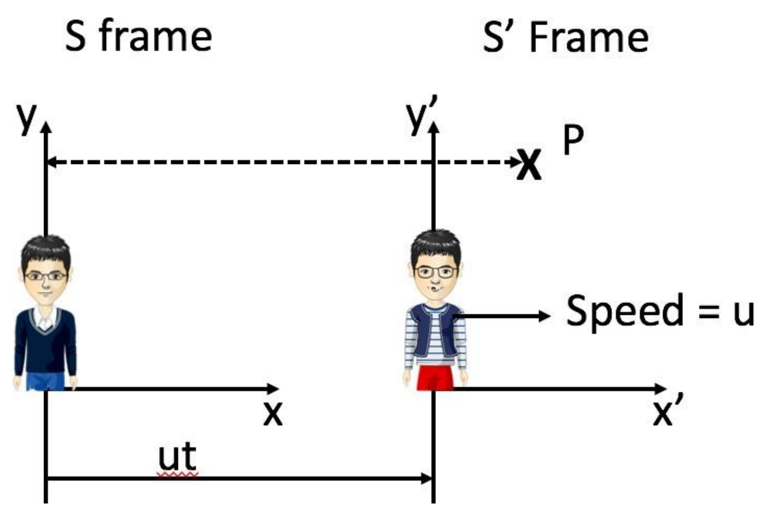
The Galilean Transformation rules fail when we try to investigate relativistic motions (motions with speed very close to the speed of light).

This is because it may violate Einstein's postulate that the speed of light is invariant in all inertial reference frames.

To resolve the problem, we have to introduce a new kind of transformation rules named the **Lorentz Transformation rules**.

3.2 Lorentz Length Transformation

(a) Lorentz Transformation Rules of Length



We can relate the spatial coordinates x and x' in S frame and S' frame by the Lorentz Transformation equations which are as follow:

$$x' = \gamma(x - ut) \quad \text{and} \quad x = \gamma(x' + ut')$$

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3.3 Lorentz Time Transformation

(a) Lorentz Transformation Rules of Time

Using the equations of Lorentz Transformation of Length, we can retrieve the 2 equations governing the relations between t and t' as follow:

$$t = \gamma\left(t' + \frac{ux'}{c^2}\right) \quad \text{and} \quad t' = \gamma\left(t - \frac{ux}{c^2}\right)$$

It is notable that the Lorentz Transformation equations reduce to the usual Galilean Transformation rules when u << c (i.e. Non-relativistic).

These equations are very useful in our discussion on Relativity.

Key Terms

Approximation	估算	P.3	Corresponding	對應	P.4
Einstein	愛因斯坦	P.2	Imaginary	假想	P.4
Invariant	不相干	P.2	Lorentz Factor	勞倫茲因數	P.4
Lorentz Transformation	勞倫茲變換	P.3	Proper length	在相對於觀察者而言是靜止的座標系中所量到的長度	P.4
Spacetime	時空	P.4	Violate	違反	P.2

Chapter Exercise

Multiple Choice Questions

1. What the proper time pf moving frame means?
   - A. The time measured for moving frame by the rest frame
   - B. The time measured for rest frame by the moving frame
   - C. The time measured for moving frame by the moving frame
   - D. The time measured for rest frame by the rest frame

2. For the length measurement in front of the rocket, which is head-on moving towards the observer(rest) in speed $\frac{c}{2}$. Which following statement is correct?
   - A. The length of rocket contracted to 1/2.
   - B. The length of rocket extended to 2.
   - C. The length of rocket extended to 1.15.
   - D. The length doesn't change.

3. A and B can be observed by each other. When A is moving in very high speed, e.g close to $\frac{c}{2}$, B is staying at rest. Which following is the best statement?
   - A. Length contraction and time dilation occurs in A and B.
   - B. Length contraction occurs in A while and time dilation occurs in B.
   - C. Length contraction occurs in B while and time dilation occurs in A.
   - D. We don't know as the observer is unknow.

4. What is the improve by using the Lorentz transformation instead of using Galilean Transformation
   - A. The observable speed must slower than speed of light. Except light.
   - B. The speed of an object will change as the observer changed.
   - C. The observable speed is different in different observers.
   - D. All of above are not correct.

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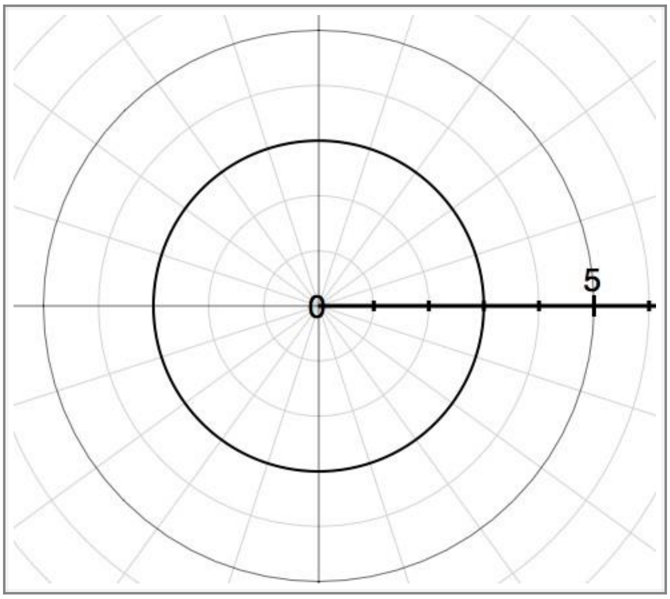
Short Questions

1. Let us consider a particle moving in the polar coordinate system.



The trajectory of the particle is given by $r = 3$, which is a circle of radius 3, centred at the origin. In an S' frame, which is rotating about the origin, the particle moves at an angular speed of $\omega = 1 \times 10^8$ rad / s.

(a) Write down the 2 postulates of Einstein's theory of Special Relativity.

(b) What is the tangential speed of the particle, as seen from the S' frame?

(c) It is known that the S' frame completes one cycle of rotation in 10 seconds. The S' frame is also at a radius 3 away from the origin.

I. Find the angular speed of the S' frame.

II. Suppose we have another S frame of radius 3 which is at rest. What would bethe tangential speed of the **particle?** Use **Galilean Transformation** in this question.

III. Is your answer in (ii) physically correct? Explain your answer.

IV. What **Transformation** rules should we use to tackle this problem if we want toget physical answer?

V. Complete the summary below:

Galilean Transformation will fail when we try to deal with motions with speed very close to the speed of______________. Therefore, we say that Galilean transformation is only a good approximation for_________-_________________________motions.

Structured Questions

[Question 1]

Consider 2 inertial reference frames S and S'. S is at rest while S' is moving at a uniform speed u relative to frame S. Suppose there are 2 events, A and B, having spacetime coordinates (t_1, x_1) and (t_2, x_2) respectively in frame S. A and B happen at the **same spatial position** in frame S.

a) Write down the relation between x_1 and x_2. (Hint : What does it mean by same spatial position?)

b) Find the corresponding time coordinates t_1' and t_2' of event A and B in frame S' using Lorentz Transformation of Time. Hence find the ratio of $t_1' : t_2'$ in terms of t_1, t_2, u and x_1.

c) Suppose that event A and event B happen **at the same time** as observed from an observer in frame S'. What is the ratio of $t_1' : t_2'$? Write the the **actual** ratio.

d) Using (c), or otherwise, show that:

$$u = \frac{c^2(t_2 - t_1)}{2x}$$

e) We define a new quantity, called the **spacetime interval**, as

$$\Delta s^2 = -(c\Delta t)^2 + \Delta x^2$$

where Δt and Δx are, respectively, the differences in temporal and spatial separation between the 2 events.

(I) Show that the spacetime interval between event A and B in the S frame is

$$\Delta s^2 = -c^2(t_2 - t_1)^2$$

(II) Show that the corresponding spatial coordinates of events A and B are

$$x'_A = \gamma(x - ut_1) \quad \text{and} \quad x'_B = \gamma(x - ut_2)$$

Hence show that

$$x'_B - x'_A = \gamma u(t_1 - t_2)$$

(III) Show that the spacetime interval between event A and B in the S' frame is

$$\Delta s'^2 = 0$$

(**Hint** : You may find it useful to express the Lorentz factor in the form of

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

(IV) What can you conclude about Δs^2 and $\Delta s'^2$?

[Question 2]

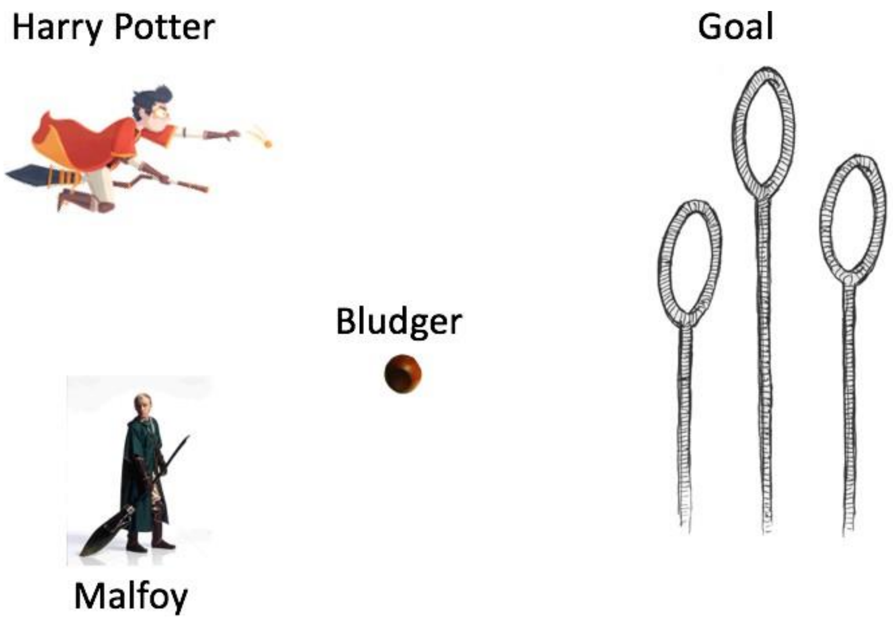
Nowadays, many technologies is applied the effects of the relativity. Can you give out some examples that the special relativity is useful in our daily life and explain briefly, in term of what you learnt in these chapters, how they works.

[Question 4]

Complete the steps for the derivation of the (i) Lorentz Transformation of Time and (ii) Lorentz Transformation of Length, such that give out the equation:

$$t = \gamma\left(t' + \frac{ux'}{c^2}\right) \qquad \text{and} \qquad x' = \gamma(x - ut)$$

[Question 5]- [The Quidditch Cup]



In the novel series "Harry Potter", there is a kind of competition called the "Quidditch". In each match, each team has to hit a **bludger** to the goal in order to score points. Each goal corresponds to 10 points. The team which gets 150 points first wins the match. In each team, there is a **seeker** whose major goal is to catch the **golden snitch** which is worth 200 points. Once the seeker gets the snitch, the match ends immediately with the winning team as that which the seeker belongs to.

In a match, Gryffindor and Slytherin are matching against each other. Near to the end of race, Gryffindor gets 50 points while Slytherin gets 140 points.

Now, Malfoy is at rest while Harry Potter is moving horizontally at a speed u relative to Malfoy, chasing the snitch. At the same time, one of Malfoy's teammate hits the bludger and it is flying towards the goal.

Define the following events: (A) Harry Potter catches the snitch, and (B) The bludger reaches the goal, with the spacetime coordinates (t_1', x_1') and (t_2', x_2') in Harry's frame. In Harry's frame, events A and B happen at the same time.

a) What is the relationship between t_1' and t_2'?

b) Show that the corresponding time coordinates of the 2 events in Malfoy's frame are

$$t_1 = \gamma\left(t_1' + \frac{ux_1'}{c^2}\right) \quad \text{and} \quad t_2 = \gamma\left(t_1' + \frac{ux_2'}{c^2}\right)$$

c) Compute t2 − t2 and express your answer in terms of x1′, x2′ and

d) If Malfoy wants his team to

I. Win the match, what should be the relationship between x_1' and x_2'? In this case, would Harry sees the snitch in front of the goal, above the goal or beyond the goal?

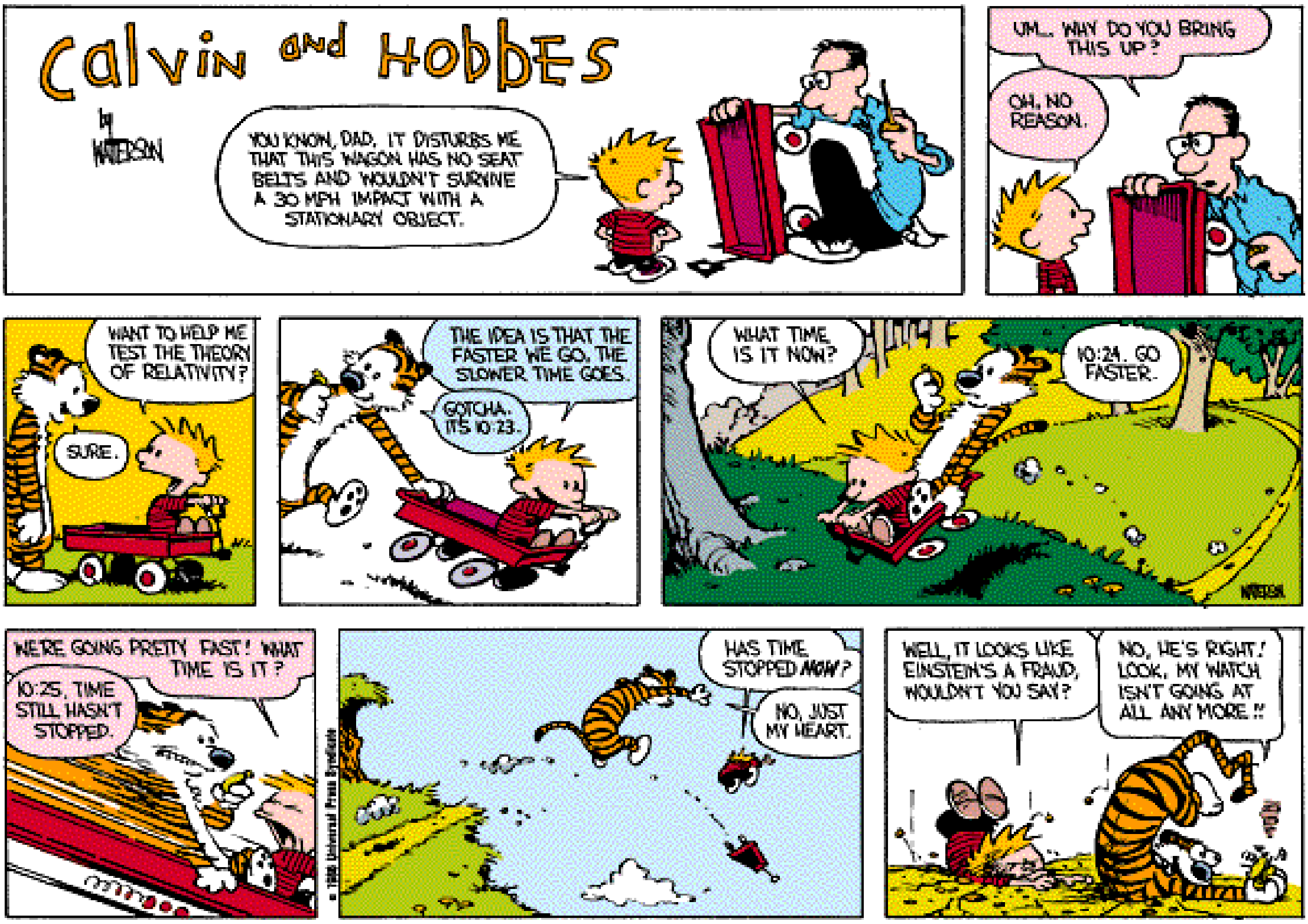
II. Lose the match, what should be the relationship between x_1' and x_2'? In this case, Would Harry sees the snitch in front of the goal, above the goal or beyond the goal?

III. Get a draw in the match, what would be the relationship between x_1' and x_2'? In this case, would Harry sees the snitch in front of the goal, above the goal or beyond the goal

THE END

Chapter 4 – More about Lorentz Transformation

Chapter Starters…



In the above comic, Calvin tries to test the theory of relativity using his **wagon** (四輪車). Try to answer the following questions to see if you still remember the special relativistic effect on time and lengths you have learnt in **Chapter 2**.

(a) In the comic, Calvin tries to increase his wagon's speed to 30 **mph** (**miles per hour**). Given that 1 mile is about 1.6×10^3 m. Express 30 **mph** in terms of **meter per second**.

(b) Let's suppose that Calvin also carries a clock during his drive. State whether there will be any difference between Calvin's clock and Hobbes's (the tiger) clock. Explain your answer.

(c) If there is an observer standing on the ground at rest and he carries a clock to measure the time used for Calvin to complete the whole journey, whose clock (Hobbes' clock or the stationary observer's clock) would measure a longer time?

(d) At the end of the comic, Calvin says that Einstein is a **fraud** (騙子) because time **HAS NOT** slowed down even though he and Hobbes are going faster. Do you agree? Can you explain what is wrong in his experiment?

4.1 – Differentiation at a First Glance

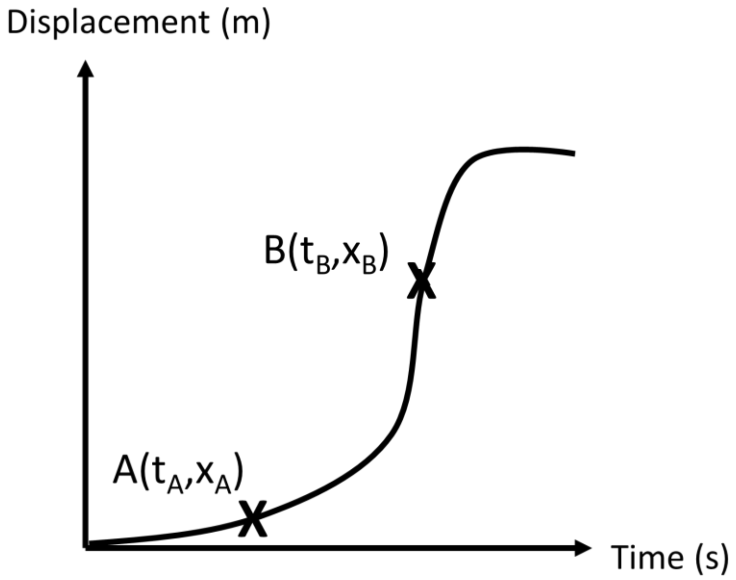
In this section, we will try to illustrate the idea of differentiation. We will also try to show the relationship of differentiation with velocity and acceleration.

Recall in Chapter 1, we say that velocity is defined by:

$$\text{Velocity} = \frac{\Delta\text{Displacement}}{\Delta Time}$$

where $\Delta\text{Displacement}$ and $\Delta Time$ are the **change in displacement** (位移改變) and **change in time** respectively.

Now, consider the following **Displacement – Time Graph** (位移時間圖) of a **moving point object** :

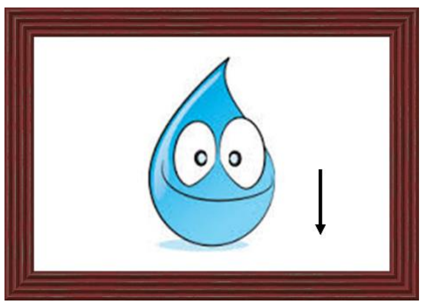


If we ask what is the **average velocity** (平均速度) between point **B** and **C**, then the answer would just be :

$$\mathrm{v_{avg}} = \frac{x_B - x_A}{t_B - t_A}$$

But if we ask for the **instantaneous velocity** (瞬間速度) of the object at point **B**, what would it be?

The **situation** (情況) is like we take a photo of a falling **water droplet** (水滴). **Obviously** (顯然地), we know that the droplet is moving downwards with a certain value of speed, but in the photo, it is **NOT moving**, so what would be its **instantaneous velocity**? Should we say that it is **instantaneously at rest** (瞬間靜止)?



Of course it is NOT! But how can we **persuade** (說服) ourselves mathematically?

Historical Facts…

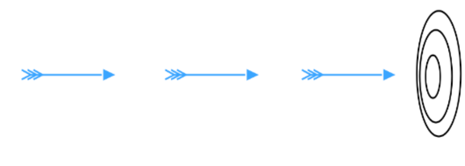
2nd Mathematical Crisis – The "Unmoving Arrow" (二次數學危機 – 飛矢不動)

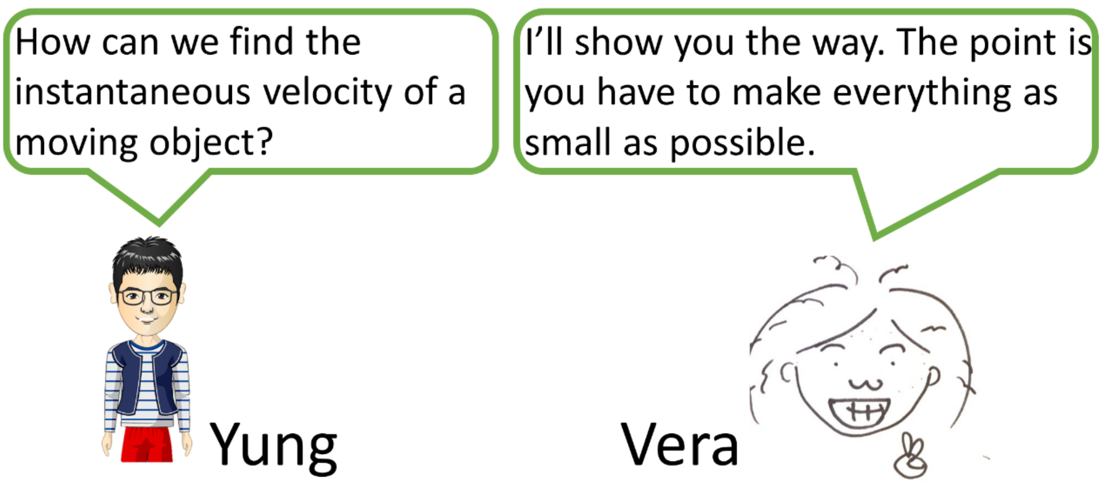
Ancient Greek **philosopher** (哲學家) **Zeno of Elea** (芝諾) once **proposed** (提出) a **paradox** (悖論) "The arrow paradox" which is quite related to the 2nd Mathematical Crisis. Here is the paradox :

One day, Zeno was walking together with his students while he suddenly started a conversation with them.

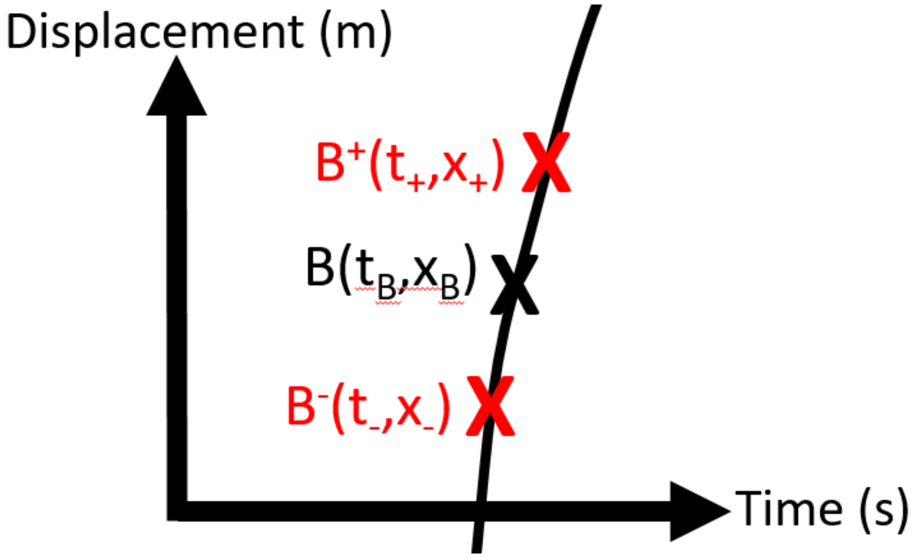
Zeno :	Is a **shot arrow** (射出的箭) **moving** or **not moving**?
Students :	The arrow must be, needless to say, **moving**.
Zeno :	True, in every people's eyes, the arrow is moving. However, does the arrow have its position in **every single instant** (每一瞬間)?
Students :	Yes, teacher.
Zeno :	In every of such instant, does the arrow **occupy** (佔有) the same **space** (空間) and **volume** (體積)?
Students :	Yes, teacher.
Zeno :	So, in one of these instants, is the arrow **moving** or **not moving**?
Students :	**Not moving**, teacher.
Zeno :	In **one instant**, the arrow is **not moving**, so how about the **other instants**?
Students :	The arrow is also **not moving** in the other instants.
Zeno :	So, can we **conclude** (結論) that a **shot arrow** is **not moving**?

The paradox is, a **shot arrow** is both **moving** and **not moving**! This is **similar** (相似) to that of the photo of a **falling water droplet**. This paradox, **fortunately** (幸運地), is finally solved by introducing the idea of **differentiation** (微分) by **Isaac Newton** (艾薩・牛頓) and **Gottfried Wilhelm Leibniz** (哥特佛萊德・萊布尼茲).





We **magnify** (放大) the original graph around point **B**. Very near to the point **B** we **define** (定義) two points $\mathbf{B}^+$ and $\mathbf{B}^-$:



The **slope** (斜率) and hence the **average velocity** between the points $\mathbf{B}^+$ and $\mathbf{B}^-$ will be :

$$\overline{v_{avg}} = \frac{x_+ - x_-}{t_+ - t_-}$$

If we further **magnify** the graph and push the two points $\mathbf{B}^+$ and $\mathbf{B}^-$ closer and closer to the point B, we will be able to get the "**slope**" and thus the **instantaneous velocity** of the object at point **B**.

This is the idea of **differentiation** (微分). To formally illustrate this idea using mathematics, we can try the following approach :

Suppose that the **displacement s** of the object at **time t** can be described by the **function** (函數) :

$$\mathrm{s} = \mathrm{f(t)}$$

The **displacement** of the object at **time** $\mathrm{t} = \mathrm{t_B}$ is $\mathrm{s} = \mathrm{s_B} = f(t_B)$.

After a **sufficiently short** (足夠地短的) time $\Delta\mathrm{t}$, the **displacement** of the object becomes $\mathrm{s} = \mathrm{s_{B+\Delta t}} = f(t_B + \Delta t)$.

Then, the **instantaneous velocity** of the object at point B will be :

$$\overline{v_B} = \frac{f(t_B + \Delta t) - f(t_B)}{(t_B + \Delta t) - t_B} = \frac{f(t_B + \Delta t) - f(t_B)}{\Delta t}$$

If we force Δt to become very close to "0", we will make the right hand side of the function become a **limit function** (極值函數). [Note : You can learn more about limits in the **Limit** Chapter.]

$$\overline{v_B} = \lim_{\Delta t \to 0} \frac{f(t_B + \Delta t) - f(t_B)}{\Delta t}$$

This **limit function** is actually called **differentiation by first principle** (從基本原理求導數), and the above function can be written as :

$$\overline{v_B} = \lim_{\Delta t \to 0} \frac{f(t_B + \Delta t) - f(t_B)}{\Delta t} = \frac{df(t)}{dt}$$

You can learn more about **differentiation** in the **Differentiation** Chapter. Here, we will only list some of the important rules and results you may find useful in doing exercises.

Mathematical Tools…

Important Results and manipulation in Differentiation (微分的重要結果及方法)

IMPORTANT RESULTS

① $\frac{d}{dx}(c) = 0 \quad (\text{c is a constant})$

② $\frac{d}{dx}(x^n) = \text{n}x^{n-1} \quad (n \geq 1)$

③ $\frac{d}{dx}(e^x) = e^x \quad (e \text{ is the natural number})$

④ $\frac{d}{dx}(\ln(x)) = \frac{1}{x} \quad (ln \text{ is the natural log})$

⑤ $\frac{d}{dx}(\sin(x)) = \cos(x)$

⑥ $\frac{d}{dx}(\cos(x)) = -\sin(x)$

⑦ $\frac{d}{dx}(\tan(x)) = \sec^2(x)$

⑧ $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$

⑨ $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$

⑩ $\frac{d}{dx}(\cot(x)) = -\csc^2(x)$

IMPORTANT MANIPULATIONS

① $$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$$

② $$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x)) \quad (c \text{ is a constant})$$

③ Product Rule :

$$\frac{d}{dx}(f(x)\times g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$

④ Quotient Rule :

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{g(x)^2}$$

⑤ Chain Rule :

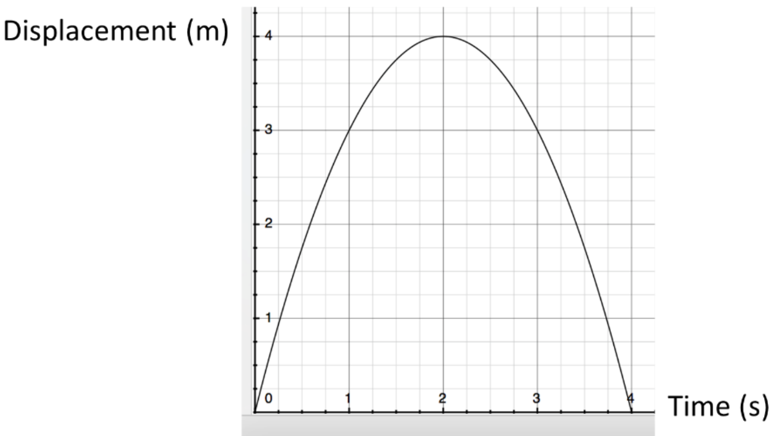
$$\frac{d}{dx}(f(g(x)) = \frac{df(g)}{dg}\times\frac{dg(x)}{dx}$$

Example 4.1

Consider the following **displacement-time graph** of an object. Its **trajectory** (軌跡) can be described by the function :

$$f(t) = -t^2 + 4t$$

where **t** is the **time of travelling**.



(a) What is the **average velocity** of the object from **t = 0** to **4s**?

(b) Show, by **first principle**, that the **instantaneous velocity v(t)** of the object at any **time t** is :

$$v(t) = -2t + 4$$

(c) Find the time when the **instantaneous velocity** of the object is 0.

[Solutions]

(a) What is the **average velocity** of the object from **t = 0** to **4s**?

[Sol] From **t = 0** to **4s**, the **total displacement** of the object is **0**. Hence, the **average velocity** of the object is also **0**.

(b) Show, by **first principle**, that the **instantaneous velocity v(t)** of the object at any **time t** is :

$$\mathrm{v(t)} = -2t + 4$$

[Sol] By First Principle, we have

$$\begin{aligned}
\mathrm{v(t)} &= \lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t} \\
&= \lim_{\Delta t\to 0}\frac{[-(t+\Delta t)^2+4(t+\Delta t)]-(-t^2+4t)}{\Delta t} \\
&= \lim_{\Delta t\to 0}\frac{[-t^2-2t\Delta t+\Delta t^2+4t+4\Delta t]+t^2-4t}{\Delta t} \\
&= \lim_{\Delta t\to 0}\frac{-2t\Delta t+\Delta t^2+4\Delta t}{\Delta t} \\
&= \lim_{\Delta t\to 0}(-2t+\Delta t+4) \\
&= -2t+0+4 \\
&= -2t+4
\end{aligned}$$

(c) Find the time when the **instantaneous velocity** of the object is 0.

[Sol] The instantaneous velocity of the object at any time t is given by :

$$\mathrm{v(t)} = -2t + 4$$

So we put v(t) = 0 and hence we solve the equation :

$$0 = -2t + 4$$

to get **t = 2**

Challenge 4.1

1. Evaluate the following limits :

(a) $\lim\limits_{h\to 0}\frac{h+x-x}{h}$ (b) $\lim\limits_{h\to 0}\frac{(h+x)^3-x^3}{h}$ (c) $\lim\limits_{h\to 0}\frac{\sin(h+x)-\sin(x)}{h}$

2. Consider a moving object with its **displacement s(t)** described by the function :

$$\mathrm{s(t)} = \mathrm{t}^3 - t + \sin(t)$$

where **t** is the **time of travel**.

(a) Using the results of **Question 1**, or otherwise, find the function **v(t)** which describes the **instantaneous velocity** of the object at any **time t**.

(b) What is the **velocity** of the object at **time t = 0**?

Spare some time and think a bit more…

- In **Question 2**, if you **differentiate s(t)** with respect to **time t** by **2 times**, what would you get?
- **Sketch** (繪畫) the graphs of **y = sin (x)**, **y = x³** and **y = – x**.

4.2 – Lorentz Transformation of Velocity

In this section, we will **derive** (推導) the equations of Lorentz Transformation of Velocity.

Recall in Chapter 3, that the **Lorentz Transformation** (羅倫茲變換) of Lengths and Time are given by the equations :

$x' = \gamma(x - ut)$	$x = \gamma(x' + ut')$
$t' = \gamma(t - \frac{ux}{c^2})$	$t = \gamma(t' + \frac{ux'}{c^2})$

Suppose there is an object moving at **velocity v** in a **rest frame S**. What would be its **velocity v'** in a **moving inertial reference frame S'**?

The solution is rather simple. In the **previous section (Section 4.1)**, we say that the **velocity** of an object is defined by :

$$v = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \frac{d}{dt}(f(t))$$

In the **relativistic view**, the **velocity v'** in the **moving frame S'** can be defined as :

$$v' = \frac{\Delta \text{Displacement in } S'frame}{\Delta Time\ \ in\ \ S'frame} = \frac{dx'}{dt'}$$

Now, from the formerly derived **Lorentz Transformation of Lengths and Time** equations, we have :

$x' = \gamma(x - ut)$ $dx' = d[\gamma(x - ut)]$ $dx' = \gamma(dx - udt)$	$t' = \gamma(t - \frac{ux}{c^2})$ $dt' = d\left[\gamma\left(t - \frac{ux}{c^2}\right)\right]$ $dt' = \gamma(dt - \frac{udx}{c^2})$

So at the end we get :

$$v' = \frac{dx'}{dt'} = \frac{\gamma(dx - udt)}{\gamma(dt - \frac{udx}{c^2})} = \frac{dx - udt}{dt - \frac{udx}{c^2}} = \frac{\frac{dx}{dt} - u}{1 - \frac{u}{c^2} \times \frac{dx}{dt}} = \frac{v - u}{1 - \frac{uv}{c^2}}$$

The above equation is called the **Lorentz Transformation of Velocity**.

Watch Out…

Be careful when you read the **Lorentz Transformation of Velocity**.

(i) The **"v"** is the **velocity** of the object in the **rest inertial reference frame S**.

(ii) The **"u"** is the **velocity** of the **moving inertial frame S'** relative to the **rest inertial reference frame S**.

(iii) The **"v'"** is the **velocity** of the object in the **moving inertial frame S'**.

Example 4.2

A point object is moving at a **speed** of **0.5 c** (c is the speed of light) in a **rest inertial reference frame S**. Another **frame S'** is moving at a **speed** of **0.8 c** relative to the **S frame**.

(a) What is the **velocity v'** of the object in the **S' frame**? Use **Lorentz Transformation** in this question.

(b) What will be the **velocity v''** of the object if we use **Galilean Transformation**?

[Solutions]

(a) What is the **velocity v'** of the object in the **S' frame**? Use **Lorentz Transformation** in this question.

[Sol] Using **Lorentz Transformation**, we have

$$v' = \frac{v - u}{1 - \frac{uv}{c^2}} = \frac{0.5c - 0.8c}{1 - \frac{(0.5c)(0.8c)}{c^2}} = \frac{-0.3c}{0.6} = \mathbf{-0.5c}$$

(b) What will be the **velocity v''** of the object if we use **Galilean Transformation**?

[Sol] Using **Galilean Transformation**, we have **v''** = 0.5c – 0.8c = **– 0.3c**

Challenge 4.2

2 rockets **A** and **B** are travelling in space. According to an **observer** on the Earth, the **velocity** of **A** and **B** are $u_A = 0.3\ c$ and $u_B = -0.7\ c$ respectively. Find the velocity of **rocket B** with respect to **rocket A**.

Summary

Key Points

4.1 Differentiation at a first glance

- **Instantaneous velocity** is actually the **first derivative** of **displacement** defined by :

$$v = \frac{ds}{dt}$$

- **Differentiation** is a kind of **limit function** in which we try to find the **"slope"** of a point in a graph.
- Because of Einstein's postulate that the **speed of light** is **invariant** in all **inertial reference frames**, we have to use **Lorentz Transformation** instead of **Galilean Transformation** to find velocity of objects in different **reference frames**.

4.2 Lorentz Transformation of Velocity

- The **velocity v'** of an object in a **moving reference frame** is defined by :

$$v' = \frac{v - u}{1 - \frac{uv}{c^2}}$$

Key Terms

Average velocity 平均速度	P.2	Define 定義	P.4
Derive 推導	P.8	Differentiation 微分	P.3
Displacement 位移	P.2	Differentiation by First Principle 從基本原理求導數	P.5
Displacement-Time Graph 位移時間圖	P.2	Function 函數	P.4
Instantaneously at rest 瞬間靜止	P.2	Instantaneous velocity 瞬間速度	P.2
Instant 瞬間	P.3	Limit function 極值函數	P.4
Lorentz Transformation 羅倫茲變換	P.8	Magnify 放大	P.4
Paradox 悖論	P.3	Philosopher 哲學家	P.3
Propose 提出	P.3	Sketch 繪畫	P.8
Slope 斜率	P.4	Sufficiently 足夠地	P.4
Trajectory 軌跡	P.6	Zeno of Elea 芝諾	P.3

Check Your Concepts

1. Why do we need **differentiation**? How is it related to the **speed** of a **moving object** in a **static photo**? [**Section 4.1**]

2. Can you find the **first derivatives** of **sin (x)**, **cos (x)** and **tan (x)** from **differentiation by first principle**? [**Section 4.1**]

3. What are the **TWO equations** of **Lorentz Transformation of velocity**? [**Section 4.2**]

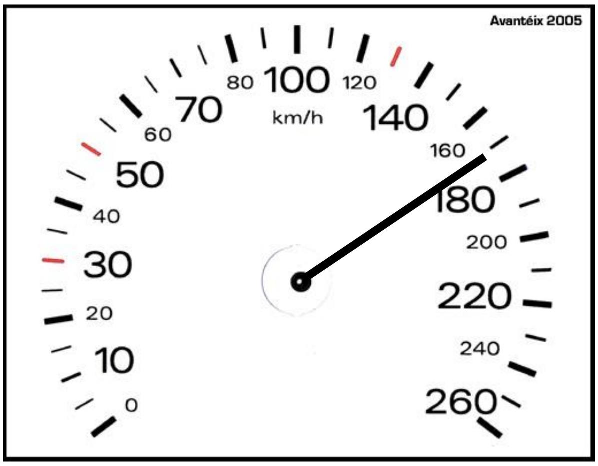
Historical Profile

Hendrik Lorentz was a Dutch physicist who shared the 1902 Nobel Prize in Physics with Pieter Zeeman for the discovery and theoretical explanation of the Zeeman effect. He also derived the transformation equations which formed the basis of the special relativity theory of Albert Einstein.

Chapter Exercise

Multiple Choice Questions

1. The following shows a **speedometer** (速率計) of a car. What is the **instantaneous velocity** of the car?



A. 44 m/s
B. 47 m/s
C. 61 m/s
D. Not enough information is given to deduce the answer.

2. A student shoots an arrow using his bow while another student takes several pictures of the arrow before it falls to the ground. Which of the following is/are correct?

(1) The velocity of the arrow is **zero** at every instant throughout its flight.
(2) If we know the function describing the trajectory of the arrow, we can find its **instantaneous velocity**.
(3) The average velocity of the arrow throughout its flight is given by :

$$\mathrm{v} = \frac{\text{Total Displacement}}{Total\ \ Time\ \ of\ \ Flight}$$

A. (2) only
B. (3) only
C. (1) and (2) only
D. (2) and (3) only

3. Which of the following shows the correct forms of finding the **derivative** of the function f(x) = x^3 + sin (x) by **first principle**?

A. $\lim_{h\to 0} \frac{[(x)^3 + \sin(x)] - [(x+h)^3 + \sin(x+h)]}{h}$

B. $\lim_{h\to 0} \frac{(x+h)^3 + \sin(x+h)}{h}$

C. $\lim_{h\to 0} \frac{[(x+h)^3 + \sin(x+h)] - [x^3 + \sin(x)]}{h}$

D. $\lim_{h\to 0} \frac{x^3 + \sin(x)}{h}$

4. Find the **first derivative** of

$$\mathrm{f(x)} = -\mathrm{x}^{\frac{3}{2}} + 2x - \tan(\frac{4}{3}x)$$

A. $-\frac{3}{2}x^{\frac{3}{2}} + 2x - \frac{4}{3}\sec^2(\frac{4}{3}x)$

B. $-\frac{3}{2}x^{\frac{1}{2}} + 2 - \frac{4}{3}\sec^2(\frac{4}{3}x)$

C. $x^{\frac{1}{2}} + 2 - \sec^2(\frac{4}{3}x)$

D. $-\frac{3}{2}x^{\frac{1}{2}} + 2 - \sec^2(\frac{4}{3}x)$

Short Questions

1. Derive the inverse Lorentz Transformation of velocity using similar steps in Section 4.2, i.e. show that

$$\mathrm{v} = \frac{\mathrm{v}' + \mathrm{u}}{1 + \frac{uv'}{c^2}}$$

where v' is the speed of the object in the moving reference frame, and u is the speed of the moving reference frame.

2. An observer **A** is at rest. Another observer **B** is moving relative to A at a speed of **0.5 c**. Now, **B** throws a ball forward at a speed of **0.5 c** relative to himself. Find the speed of the ball relative to **A** using **Lorentz transformation equations**.

Structured Questions

[Question 1] (Difficulty : ⋆)

Let's consider a rather interesting question. Suppose we have 2 photons (light particle) travelling towards each other. Each of them has a speed of c.

(a) Using Galilean Transformation, what would be the velocity of photon B as seen by photon A?

(b) Show that the velocity of photon B as seen by photon A would be equal to c if we use Lorentz Transformation.

(c) Which postulate of Einstein's theory of special relativity will Galilean Transformation violate, as shown by the calculations above?

[Question 2] (Difficulty : ⋆ ⋆)

There is a kind of particle named **Muon** with a **mean lifetime** of 2 x 10^{-6} s (as measured from the frame of reference of muon). If we neglect the effect of **time dilation**, even if it moves at the speed of light (i.e. 3 x 10^8 m/s), it can at most travel a distance of 600 m.

However, research shows that muons produced at a height of 10 km = 10^4 m above the ground can reach the ground at the end. This suggest that muons must be travelling at a very **high speed** which leads to the **time dilation effect** in **Special Relativity**.

(a) In this question we consider the muon's motion from the point of view of an **inertial rest observer** on the Earth.

(i) If the speed of the muon is **u** m/s, what would be the lifetime **t** of muon as measured from the observer on Earth? Express your answer in terms of **u**.

(ii) What would be the maximum distance travelled by the muon, as measured by the observer on Earth, according your answer in (i)?

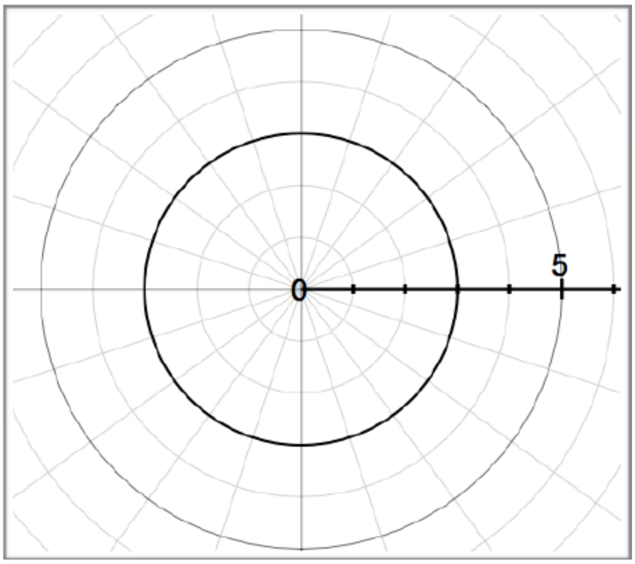
(iii) Using (ii), set up an equation to estimate the **minimum speed** of the muon if it is to be observed to travel at distance of 10 km before it disappears.

(b) In question (a) we describe the motion of muon from the point of view of an **inertial rest observer** on Earth. Now, let's consider the motion from the point of view of the muon (that is, an inertial reference frame which moves together with the muon). From that point of view, the muon particle is **at rest** while the **Earth** is **moving towards** it.

In this case, the muon's lifetime is 2×10^{-6} s in its frame, and there is no **time dilation** effect in its frame, so how can we explain why it can still reach the ground from a starting position of 10 km above the ground?

[Question 3] (Difficulty : ★ ★ ★)

Let us consider a particle moving in the polar coordinate system.



The trajectory of the particle is given by r = 3, which is a circle of radius 3, centred at the origin. In Cartesian coordinates, we can express the position of the particle by:

$$\begin{cases} x = 3\cos\theta \\ y = 3\sin\theta \end{cases}$$

where t is the time of travel.

(a) Show that the distance of any points on the trajectory described by the equations above from the origin is 3. This is the radius of the circular trajectory.

(b) Find the velocity of the particle along the (i) x-direction, and (ii) y-direction, by differentiation using First Principle.

(c) When does the particle has 0 velocity along the (i) x-direction, and (ii) y-direction?

(d) Sketch 2 curves to show the velocity of the particle along the x and y direction with respect to time t.

[Question 4] (Difficulty : ★ ★ ★)

Note : This question requires basic knowledge about **Matrix**.

The usual Lorentz Transformation equations can be written in **Matrix form**. (You can learn more about **matrix** in the extension chapter **Matrix**) as :

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma u}{c} \\ -\gamma u & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

(a) By simplifying the right hand side of the above matrix equation, show that we can obtain the usual Lorentz transformation equations :

$$x' = \gamma(x - ut) \quad \text{and} \quad t' = \gamma(t - \frac{ux}{c^2})$$

(b) Evaluate the **determinant** of the matrix :

$$\begin{pmatrix} \gamma & -\frac{\gamma u}{c} \\ -\gamma u & \gamma \end{pmatrix}$$

Need a helping hand?

To evaluate the **determinant** of a 2x2 matrix like :

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

We first write :

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

And the evaluation is just : AB – CD.

(c) Using your answer in (b), and the above matrix equation, find the **inverse Lorentz transformation equation** in **matrix form**. Verify your answer by simplifying the right hand side of your solution.

<u>Need a helping hand?</u>

① For a matrix **M** defined by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

If we can find an **inverse** $\mathbf{M^{-1}}$, then we have :

$$MM^{-1} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

② For a 2x2 matrix **M** defined by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

If the **determinant** of M = K $\neq 0$, then the **inverse** $\mathbf{M^{-1}}$ is defined by :

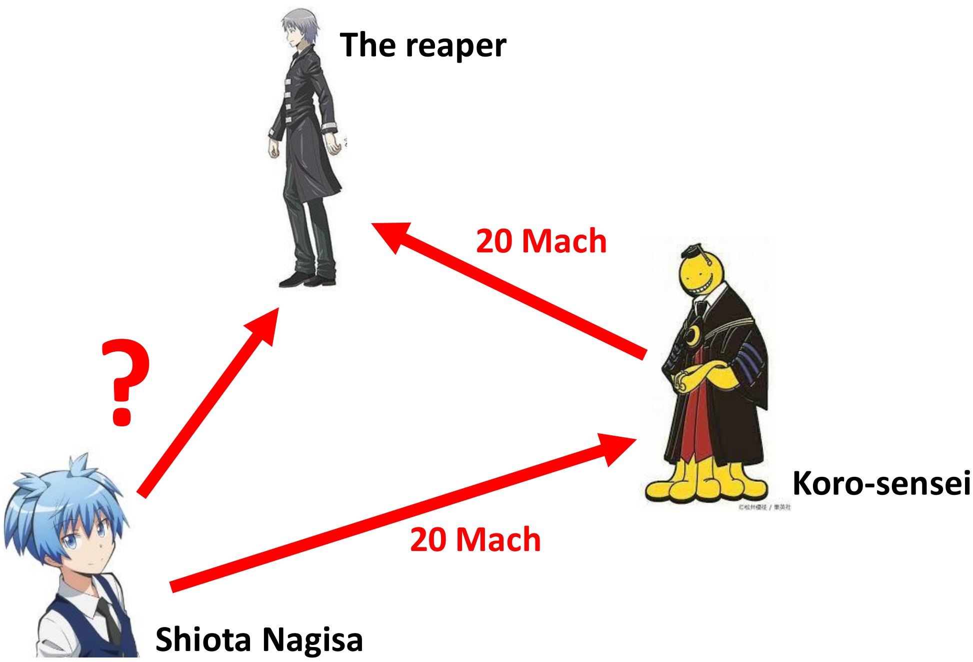
$$M^{-1} = \frac{1}{K}\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$$

(d) Show that when $c \rightarrow \infty$ (i.e. another way to say that u is much smaller than c), the above Lorentz Transformation matrix equations reduce to the usual **Galilean Transformation matrix equations** :

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -u & \gamma \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix}$$

[Question 5] (Difficulty : ★ ★ ★ ★)

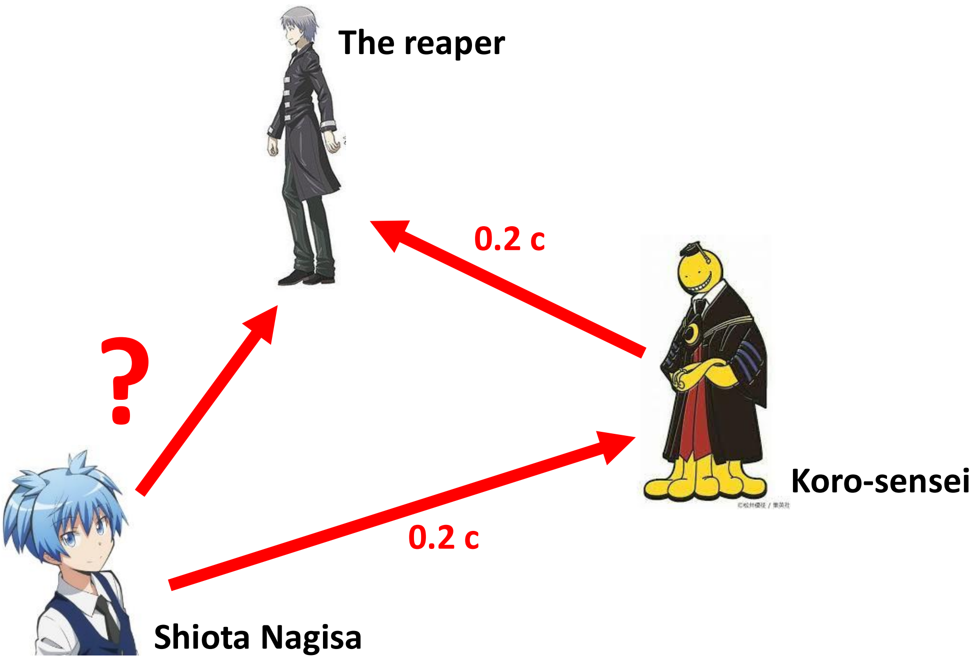
In the manga series "Assassination Classroom" (暗殺教室), a yellow monstrous-like teacher "Koro-sensei" (殺老師) can move at a speed of **20 Mach** (i.e. 20 times the speed of sound). One day, Koro-sensei wants to go to Hawaii to watch a newly-released film "Sonic Ninja". He flies to there at a speed of **20 Mach**. One of his student, Shiota Nagisa (潮田渚) observes his flight on the ground. During his flight, Koro-sensei sees another monstrous-like man "The reaper" (死神) flying pass him. From Koro-sensei's point of view, the reaper is travelling at a speed of **20 Mach**.



(a) (i) Given that the speed of sound is about 340 m/s. Express **20 Mach** in unit of **m/s**.

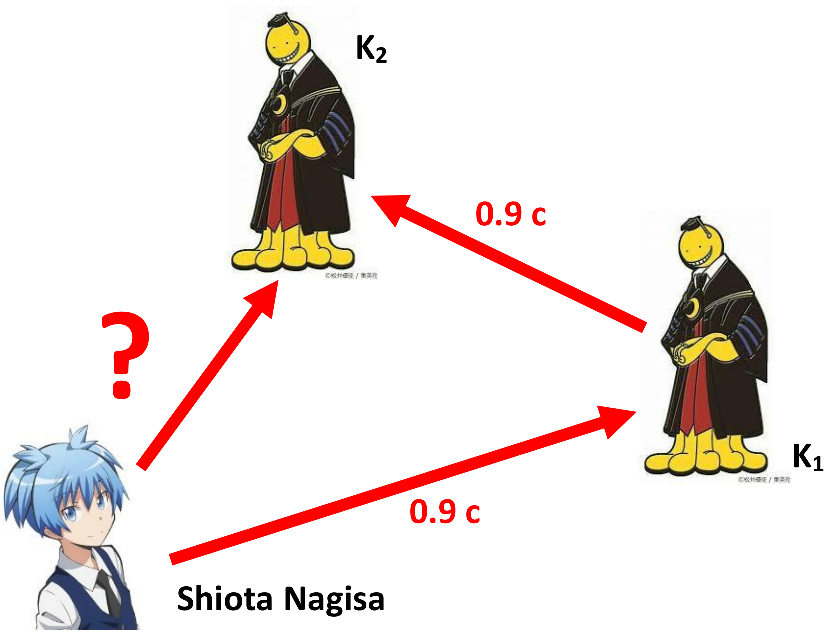
(ii) From your answer in (i), **20 Mach** is indeed much smaller than the speed of light, and thus we can use **Galilean Transformation** in our calculation. Using Galilean Transformation, find the speed of "the reaper" from the point of view of **Shiota Nagisa**.

(b) Now, we assume that after receiving certain kinds of treatment, the maximum speed of both Koro-sensei and "the reaper" increase dramatically. Now, it is known that Koro-sensei is travelling at a speed of 0.2 c (c is the speed of light). From his point of view, "the reaper" is travelling at a speed of 0.2 c.



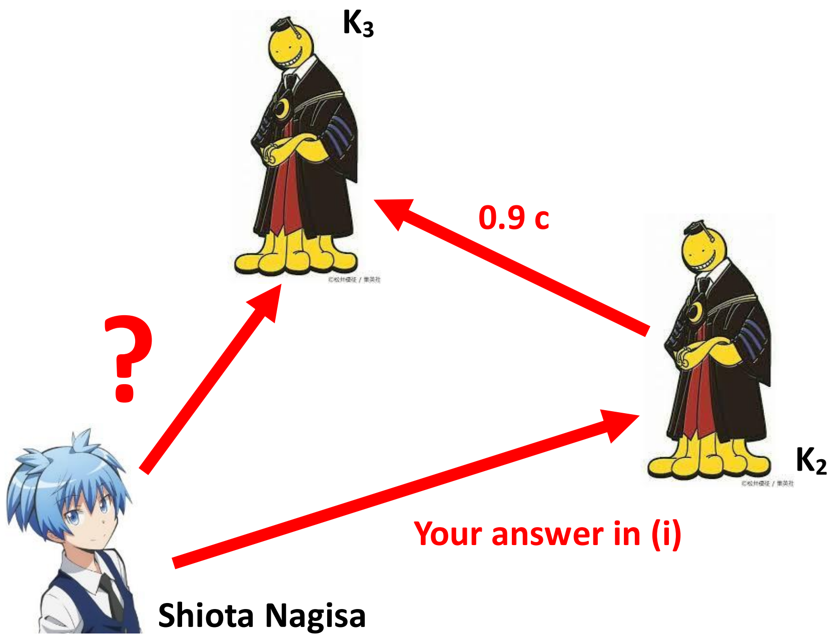
Using **Lorentz Transformation**, find the speed of "the reaper" as observed by Shiota Nagisa.

(c) Indeed, the scientist who invented **Koro-sensei** has made more **koro-sensei(s)**. Let's denote the 1st **koro-sensei** as K_1. K_1 is moving at a speed of **0.9 c** relative to **Shiota Nagisa**. From K_1's point of view, another **koro-sensei** (K_2) moves at a speed of **0.9 c** relative to him.



(i) Using **Lorentz Transformation**, find the speed of "K_2" as observed by Shiota Nagisa and express your answer in fraction.

(ii) Now, K_2 saw another **Koro-sensei** (K_3) moving at a speed of **0.9 c** relative to him.



Using **Lorentz Transformation**, find the speed of "K_3" as observed by Shiota Nagisa and express your answer in fraction.

(iii) Repeat (ii) if there is another **koro-sensei** (K_4) moving at a speed **0.9 c** relative to K_3 and express your answer in fraction.

(iv) Your answer in (i), (ii), (iii) are expressed by fractions : $\frac{A}{B}c$, $\frac{C}{D}c$ and $\frac{E}{F}c$ respectively (c is the speed of light).

(1) Find a relationship between the **numerator** and the **denominator** of these fractions.

(2) Find the value of $C \div A$ and $E \div C$, round down to the nearest integer.

(v) If there are in fact **N koro-sensei(s)** (i.e. K_1, K_2, K_3 … K_N), find an **approximation** of the speed of K_N relative to Shiota Nagisa **by using your answer in (iv)**.

[Question 6] (Difficulty : ★ ★ ★ ★ ★)

Note : This question requires basic knowledge about **Matrix**.

For simplification, neglect y-coordinate and z-coordinate in the following calculations.

Consider 2 **inertial reference frames S** and **S'** with coordinates (x , t) and (x', t') respectively, where x and t are **spatial** and **time coordinates** respectively. Initially, the origins of **S** and **S'** are at the same point.

S' is moving relative to **S** along the x-axis at a speed of **v**. Under **Lorentz Transformation**, the coordinates transformation can be obtained by the equations :

$$\begin{cases} x' = f(v)(x - vt) \\ t' = g(v)(t - m(v)x) \end{cases}$$

where f(v), g(v) and m(v) are **functions of v** to be determined later.

(a) A light signal is emitted at the origin along the **positive x-direction** in the **S-frame**.

(i) Write down an equation connecting **x** and **t** which describes the subsequent motion of the light signal.

> Need a helping hand?
>
> ① "x" is the **distance** travelled by the light signal.
>
> ② "t" is the **time of travelling** of the light signal.
>
> ③ What is the speed of the light signal?

(ii) Write down an equation connecting **x'** and **t'** which describes the subsequent motion of the light signal as seen in the **S'** frame.

> Need a helping hand?
>
> ① Something about the light signal is **unchanged ("invariant")** under one of the postulates of Special Relativity. What is it?

(iii) Using your results from (i) and (ii) and the given **Lorentz Transformation equation** to obtain **an equation** connecting **f(v)**, **g(v)** and **m(v)**. Name this equation as **(1)** (Don't worry, the final equation is kind of "ugly" =))

(b) Repeat (a) if another light signal is emitted at the origin along the **negative x-direction** in the **S-frame**. Name the final equation you obtained as **(2)**

> Need a helping hand?
> In this case, the light is moving towards the **left**, so the distance travelled would be **negative**.

(c) (i) By considering performing some **manipulations** with equations **(1)** and **(2)**, find **m(v)** in terms of **v** and **c** (c is the speed of light).

> Need a helping hand?
> We want to eliminate **f(v)** and **g(v)** to get an equation involving only **m(v)**, **v** and **c**. Observe the **similarities** between equation **(1)** and **(2)**. Which manipulation (**i.e. addition, subtraction, multiplication and division**) can help you eliminate **f(v)** and **g(v)**?

(ii) Hence, or otherwise, show that **f(v) = g(v)**.

> Need a helping hand?
> ① For the "**hence**" approach, you have to substitute your answer in (i) into either equation **(1)** or **(2)** to show the required result.
>
> ② For the "**or otherwise**" approach, you have to think of **one manipulation** upon equation **(1)** and **(2)** to eliminate **m(v)**.

(d) Using **matrix representation**, the above **Lorentz Transformation equations** can be written in the form :

$$\begin{bmatrix} x' \\ \mathrm{t}' \end{bmatrix} = f(v)\begin{bmatrix} 1 & -v \\ -\mathrm{m(v)} & 1 \end{bmatrix}\begin{bmatrix} x \\ \mathrm{t} \end{bmatrix}$$

[Note : We have proved that f(v) = g(v) in the previous question. Here we use f(v)]

When we consider the **inverse Lorentz Transformation**, the **S frame** will be moving at a speed of **– v** as seen from the **rest S' frame**.

Because of the **symmetry** along the vertical line passing through the origin, we have **m(– v) = – m(v)**, together with **f(– v) = f(v)** and **g(– v) = g(v)**.

Hence, the **inverse Lorentz Transformation** can be represented by :

$$\begin{bmatrix} x \\ \mathrm{t} \end{bmatrix} = f(v)\begin{bmatrix} 1 & v \\ \mathrm{m(v)} & 1 \end{bmatrix}\begin{bmatrix} x' \\ \mathrm{t}' \end{bmatrix}$$

Using the above **two matrix equations**, find f(v).

<u>Need a helping hand?</u>

① Substitute the 2nd matrix equation into the **right hand side** of the 1st matrix equation, then simplify the expression to obtain f(v).

② Some useful and simple manipulations of matrix :

(a) $\begin{bmatrix} A & B \\ C & D \end{bmatrix} f(x) \begin{bmatrix} E & F \\ G & H \end{bmatrix} = f(x) \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix}$

(b) $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$

(c) $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

(d) $f(x) \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} Af(x) & Bf(x) \\ Cf(x) & Df(x) \end{bmatrix}$

(e) $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$ if and only if A = E, B = F, C = G and D = H.

[Question 7] (Difficulty : ★ ★ ★ ★ ★)

In a rest inertial reference frame **S**, the coordinate system can be represented by (t , x), where t is the **time coordinate** and x is the **spatial coordinate**.

Another inertial reference frame **S'** is moving at a speed **v** along the **positive x-direction**. The coordinate system can be represented by (t' , x').

The usual **Lorentz Transformation equations** from the **S** frame to the **S'** frame are :

$$\begin{cases} ct' = \gamma(ct - \beta x) \\ x' = \gamma(x - \beta ct) \end{cases}$$

where

$$\beta = \frac{v}{c} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

(a) Using the above equations, find the **inverse Lorentz transformation equations**. (i.e. find the equation for **ct** and **x**)

(b) There is a **rod** parallel to the **x'** axis which is at rest in the **S'** frame. The coordinates of the **left end** and **right end** of the rod in the **S'** frame are (t' , x_L') and (t' , x_R') respectively.

(i) Denote the **length** of the rod as L_0. Express L_0 in terms of x_L' and x_R'.

(ii) Find the **length L** of the rod as measured in the **S** frame in terms of γ and L_0

<u>Need a helping hand?</u>

① Do you remember the **length contraction equation**?

$$L_0 = \gamma L$$

② Who measures the **proper length** now? The observer in the **S** frame or the **S'** frame?

(c) The following shows how **Paul** attempts to use **inverse Lorentz Transformation** to find the **length L** of the rod in the **S** frame :

<u>Paul's attempt</u> :

Using the **inverse Lorentz transformation equations**, we can find the x-coordinates of the 2 ends of rod in the **S** frame. The difference between the x-coordinates will be the required length **L**.

<u>Steps</u> :

① Using the equations, we have

$$\begin{cases} x_L = \gamma(x_L' + \beta ct') \\ x_R = \gamma(x_R' + \beta ct') \end{cases}$$

② Therefore, we have the required length **L** as :

$$\mathbf{L} = x_R - x_L = \gamma(x_R' - x_L') = \gamma L_0 > L_0$$

③ From the calculation, we can see that "a moving rod" **expands** instead of **contract** from a rest inertial observer point of view.

Obviously, we know that a moving rod seems to **contract** from the point of view of a **rest inertial observer**. What is **wrong**, then, in **Paul's attempt**?

(d) Now, we assumed that we put an ideal synchronized clock which co-move with the **S'** frame. The clock has a spatial coordinate of x'_{clock} in the **S'** frame.

(i) Using the **inverse Lorentz Transformation equations**, verify the statement :

*A moving clock **runs slower**.*

(ii) Repeat (i) using the **Lorentz Transformation equations**.

~ The End~

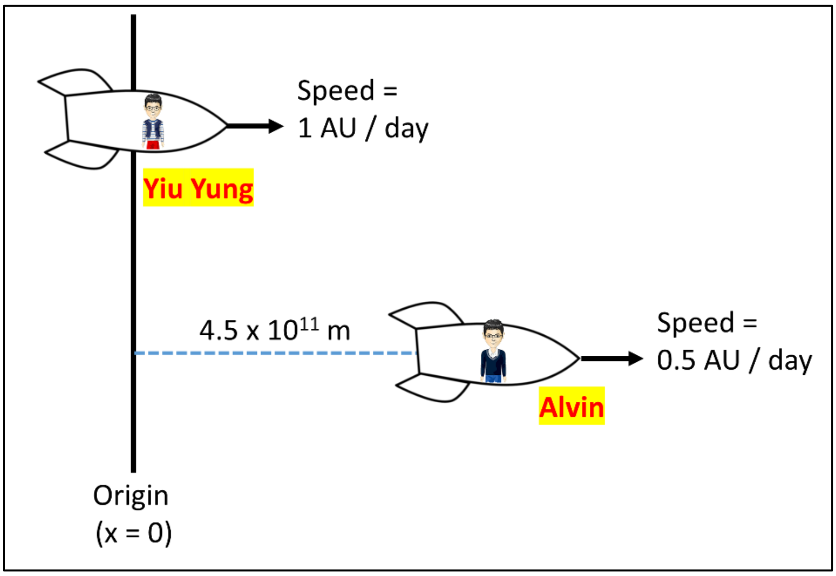
Chapter 5 – Usage of Spacetime Diagrams

Chapter Starters…

It will be very boring if we can only use mathematics to deal with problems in Special Relativity. Can we draw some kinds of pictures to have some fun?

The question is – Why CAN'T we? Of course we can draw pictures. We will introduce, in this chapter, the spacetime diagram to the readers. You can illustrate relativistic ideas and situations by making use of the spacetime diagram.

Let's first review how to make use of a distance-time graph in the following question.



Two friends, Alvin and Yiu Yung, are racing in space travelling in their spaceships. At time t = 0, Alvin is at a distance of 4.5 x 10^{11} m right from the origin (x = 0) and Yung is at the origin.

(a) One **astronomical unit** (天文單位) [AU] is defined as the **average distance** between the Sun and the Earth. Given that 1 AU = 1.5 x 10^{11} m. Find how many AU is Alvin away from the origin.

(b) Suppose Yiu Yung travels at a speed of 1 AU / day. Write down an equation relating x and t for Yiu Yung. (Hint : How far will Yiu Yung travel t days later? What does "x" represent?)

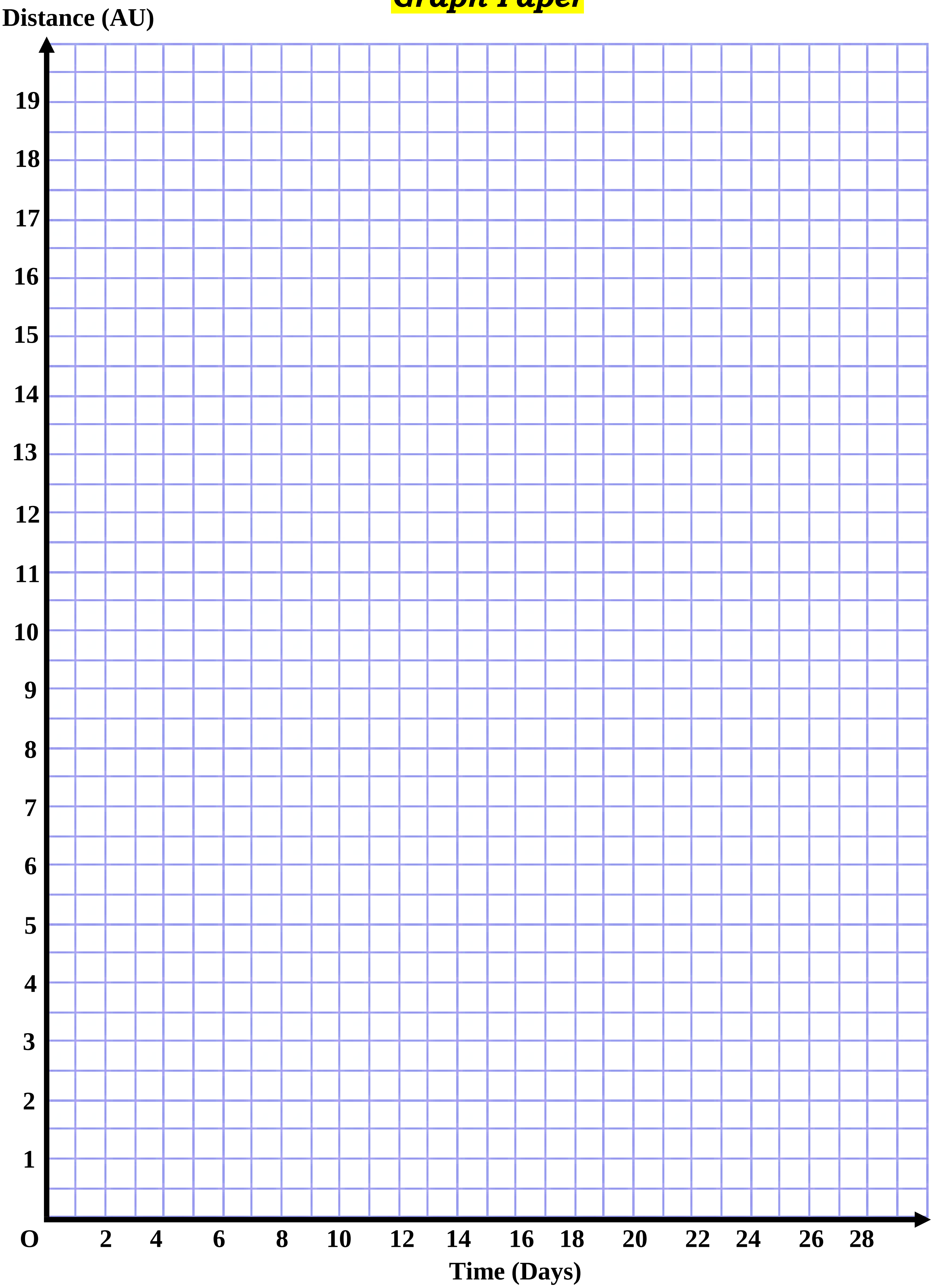
(c) Suppose Alvin travels at a speed of 0.5 AU / day. Write down an equation relating x and t for Alvin. (Hint : How far will Alvin travel t days later? Where is Alvin at t = 0?)

(d) Sketch the 2 equations of straight lines in (b) and (c) in the graph next page.

(e) Using the graph, or otherwise, find when Yiu Yung will catch up with Alvin.

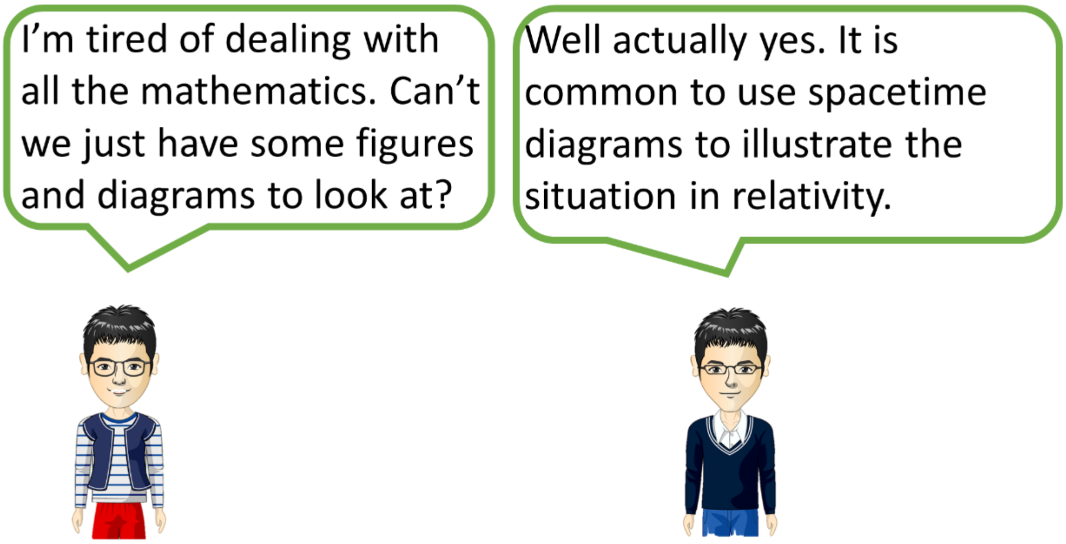
(f) After Yiu Yung catches up with Alvin, he starts to return to his original position (i.e. the origin) at a speed of 0.5 AU / day. Find when he will return to his starting position graphically.

Graph Paper


Distance (AU)
19
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
O
2 4 6 8 10 12 14 16 18 20 22 24 26 28
Time (Days)

5.1 – Introduction to Spacetime Diagram

In this section, we will introduce spacetime diagram to the readers, explaining its major features, including the axes, world line etc.



You may have already encountered some graphs like "**Distance – Time graph**" (距離時間圖), "**Displacement – Time Graph**" or so. These graphs describe the **(relative)** (相對的) position of a certain person or object.

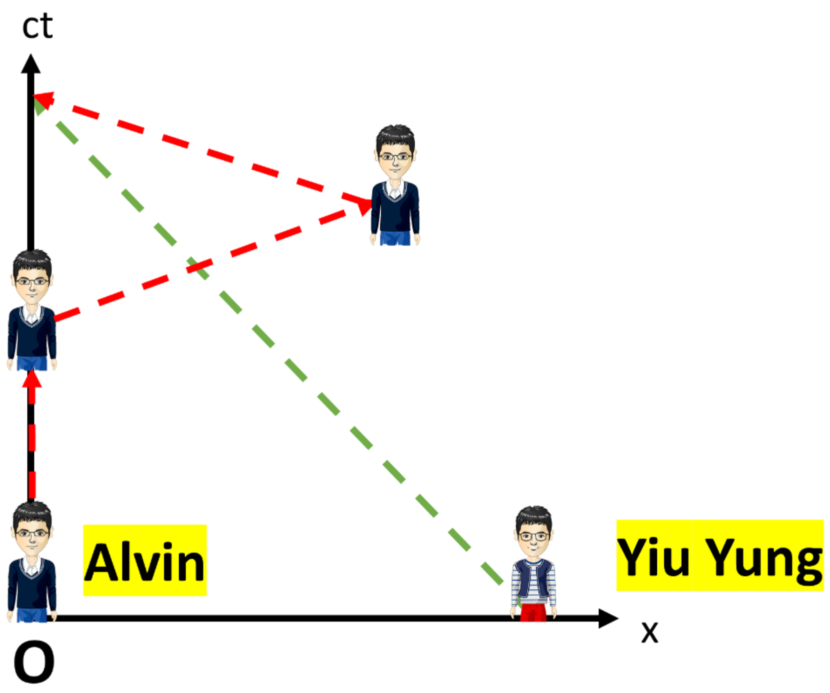
In Special Relativity we also have a kind of graph which is similar to the above mentioned graphs, which is called the **Spacetime diagram** (時空圖).

A usual spacetime diagram takes the usual format as that of **Cartesian coordinates** (直角坐標系), that is, it is formed by **2 perpendicular axes**, one **vertical** (垂直的) and one **horizontal** (水平的).

The **horizontal** axis refers to the **space axis** (空間軸)*. It is the **spatial position** (空間上的位置) of **events** (事件).

(***Note** : In general, the position of an event is **3-dimensional** (三維的) [i.e. It should have x, y and z coordinates in space]. In **most of the** discussions and problems in this set of notes, we will focus on only **1-dimensional** cases [i.e. You may regard the **space axis** as the usual **x-axis**].)

On the other hand, the **vertical axis** refers to the **time axis** (時間軸). We intentionally multiply the time axis **t** by the speed of light **c** to simplify things. [See **"Want to know More?..."** if you want to know the reason behind this act.]

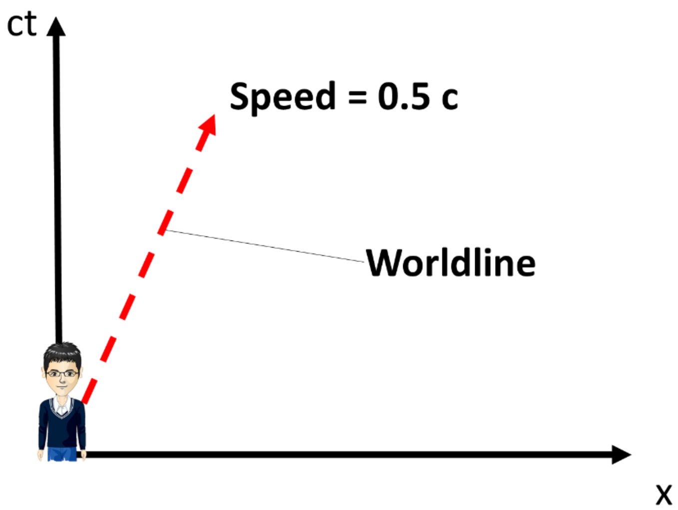


Let's **examine** (研究) the above spacetime diagram.

In the above diagram, Alvin is first **at rest** at **x = 0**. Then, he moves to the **right** (**positive-x direction**) and comes to rest. Finally, he walks back to **x = 0** at the end.

On the other hand, Yiu Yung moves towards **x = 0** at an **uniform speed**.

There are some **features** (特點) on a spacetime diagram. We shall **examine** them one by one.



Suppose Alvin is moving to the **right** at a uniform speed of 0.5 c. The equation which connects Alvin's x-coordinate and t-coordinate will be :

$$x = 0.5\, ct \quad (y = ct)$$

$$y = 2x$$

We can plot a **straight line** (red line) on the spacetime diagram to show the path or **trajectory** (軌跡) of Alvin. We say that the red line is the **world line** (世界線) of Alvin.

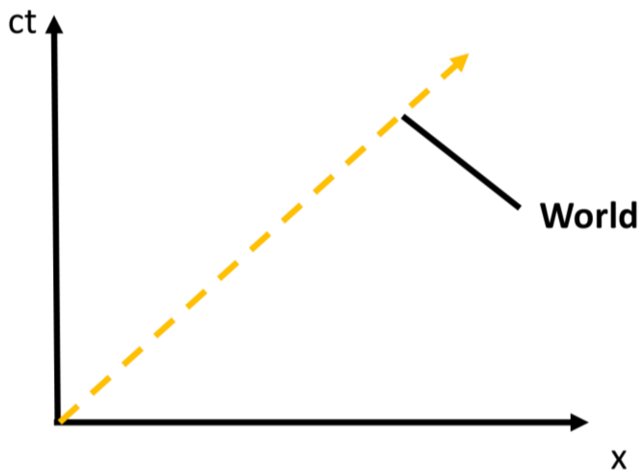
What about the **world line** of **light**?

Light is travelling at a speed of **c**, so the equation for its world line will be :

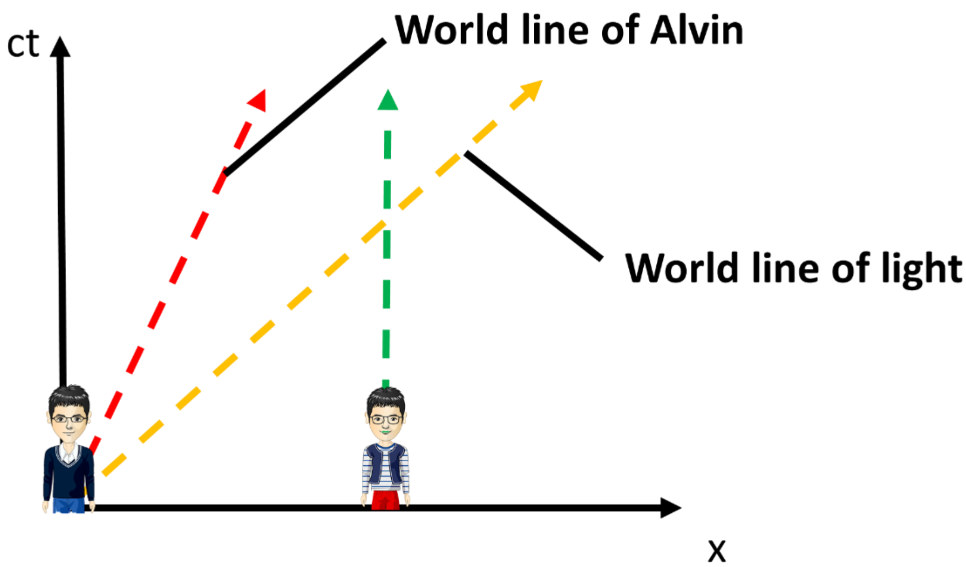
$$x = ct \quad (ct = y)$$

$$y = x$$

So it will be a **straight line** (yellow line) making an **angle 45°** with the **x-axis**.



Now, look at the following **spacetime diagram**. What does the green line represent? What is the motion of Yiu Yung?



Answer : The green line represents the **world line** of Yiu Yung. He is **at rest** somewhere right to the origin.

Want to know More?...

"t" versus "ct"...Wait! Aren't they of different dimensions? ("t" VS "ct"...它們不是不同量綱嗎?)

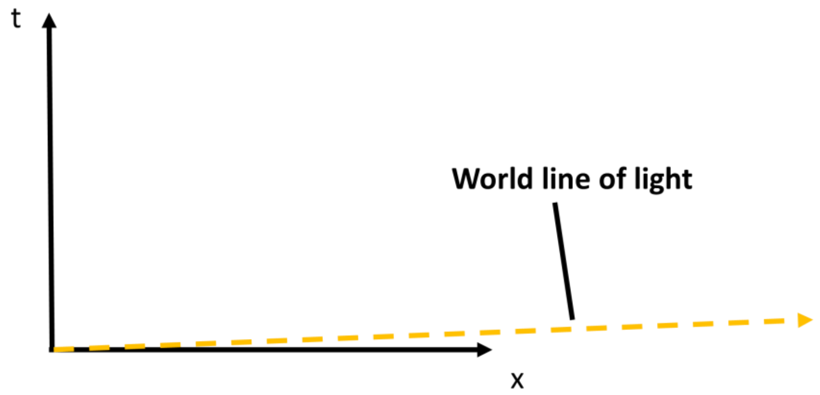
Good question. The author has also been **pondering** (思索) on this question. Right, we meant to construct a graph similar to that of a **distance – time graph**, but now **ct** would be having a **length dimension [m]** instead of a **time dimension**. So what's the point of doing this?

The point is, what is the **SI unit** of **time** and **length**? **[s]** and **[m]**.

Still remember the **world line of light** :

$$x = ct$$

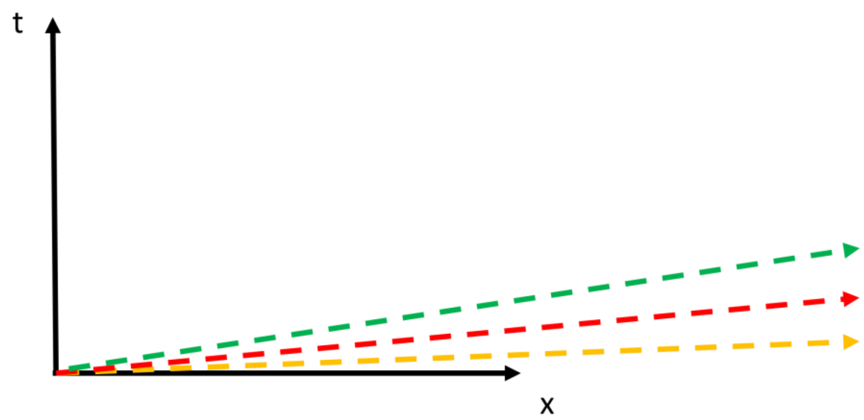
If we plot it on a **t-x diagram**, how would it look like?



It would probably look like the **yellow line** in the above diagram.

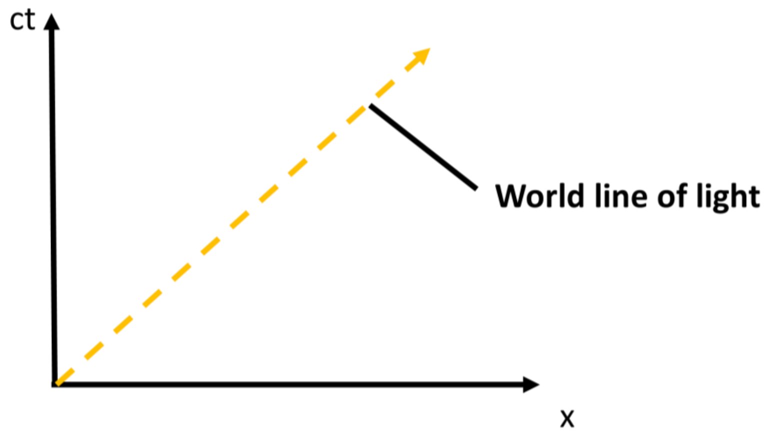
You might ask, what's the matter of this? Here's the problem : When would relativistic effect become important?

Of course when objects travel at a speed **close to the** speed of light. Let's draw the world lines of a few of these objects on the above diagram to see what will happen.



The lines would be very close to the world line of light. Can you imagine how we are going to further do **drawings** on the diagram?

To help us solve the problem, we intentionally multiply the speed of light **c** to the **time (t) axis** to make it more **convenient** (方便) in doing drawings.



Although the vertical axis **ct** now carries a **length dimension**, you should still somehow interpret it as a **time**. You can convince yourself by saying that each unit length on the **ct axis** is "**the time required for light to travel 1 m**".

Another important **result** which rises from this construction is the **calculation of spacetime interval** (時空區間) which you will learn more in **Chapter 6**.

In relativity we want some kinds of rule similar to that of **Pythagoras theorem**. Physicists soon found an expression which can be **invariant** under coordinate transformation, which is the **spacetime interval**:

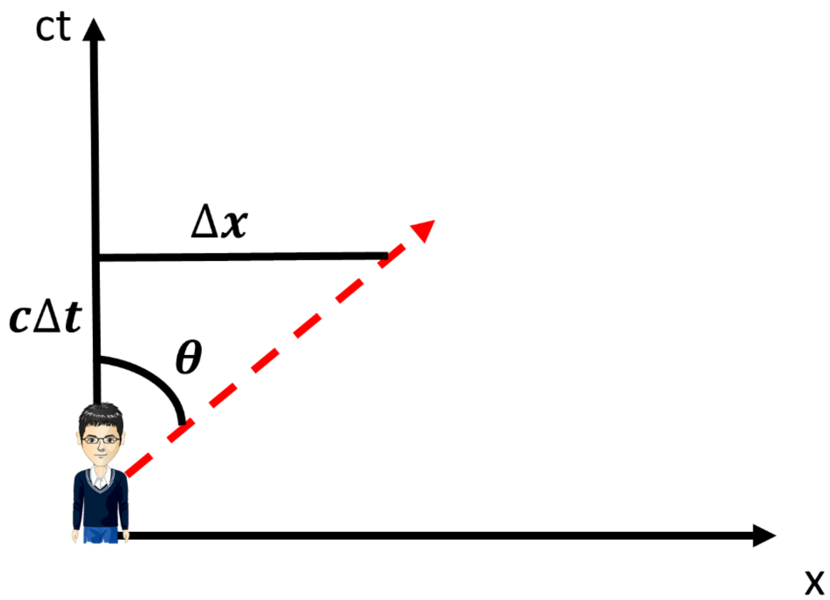
$$\Delta S^2 = -(ct)^2 + x^2 + y^2 + z^2$$

This is similar to the **Pythagoras theorem** we used to, except 2 differences:
(1) An **additional c** is multiplied to the **time coordinate**.
(2) The sign in front of the **time-coordinate** is **negative** instead of **positive**.

We will not explain this here in this Chapter. You can find out more in **Chapter 6**, in which we will formally introduce this new concept.

However, you can notice the appearance of **ct** in the above expression **suggests** the vertical axis in a spacetime diagram to be **ct** instead of **t**.

Let us now **examine** another **feature** of the spacetime diagram.



Again, the red line represents the **world line** of Alvin. What is the **angle** $\boldsymbol{\theta}$ between the world line of Alvin and the **ct-axis**?

In fact, we can find the value of the **angle** $\boldsymbol{\theta}$ by the following equation :

$$\tan\theta = \frac{\Delta x}{c\Delta t} = \frac{1}{c}\left(\frac{dx}{dt}\right) = \frac{u}{c}$$

where **u** is the **speed of Alvin**.

Up till now, we believe nothing can be **faster than the speed of light c**. So the upper limit of **u** would be **c**. Thus, we have :

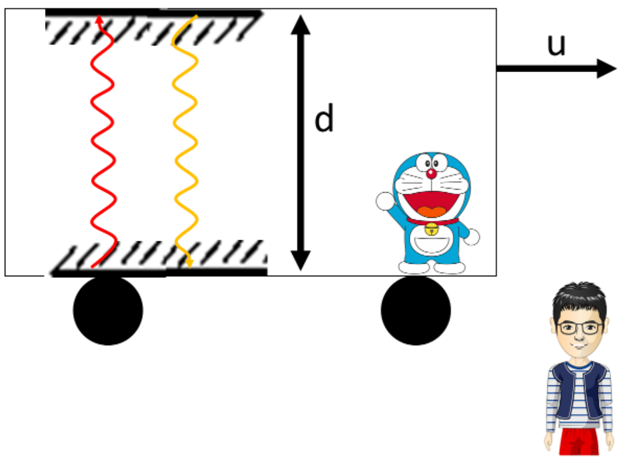
$$\tan\theta = \frac{u}{c} < \frac{c}{c} = 1$$

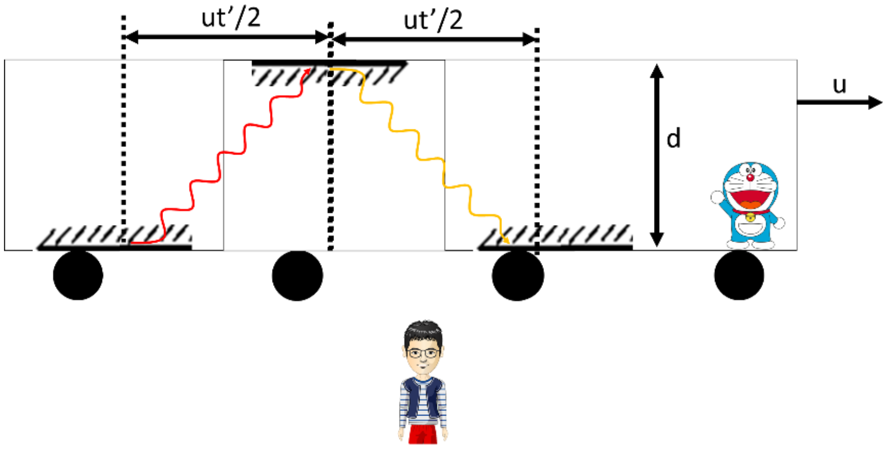
and hence we have the **angle** $\boldsymbol{\theta}$ of any object on the spacetime diagram would be smaller than **45°**.

Let's use a case that we are **familiar with** (熟悉的) to apply the spacetime diagram.

Recall in **Chapter 2**, we use the following "**light in a moving car**" case to derive the **time-dilation** equation?

Background of the case :
Inside a moving car, there are **2 mirrors**. At time t = 0, a light signal is sent from the **bottom mirror** to the **upper mirror**. After being **reflected** from the **upper mirror**, it returns to the bottom mirror. When the light signal is first sent, both **Doraemon** (on the car) and **Yiu Yung** (on the road at rest) both start a timer to count the time of travel of the light signal.





Now, how should we draw the spacetime diagram for the **events** (事件)?

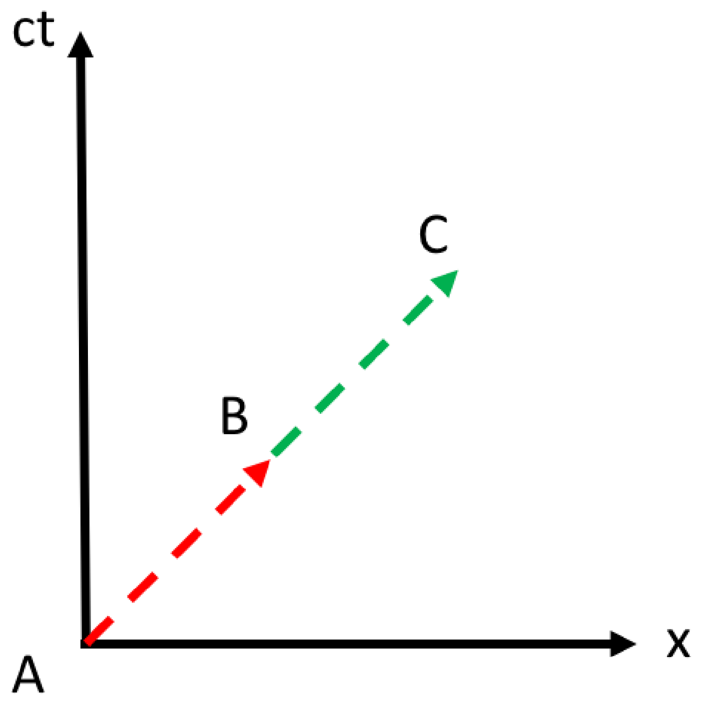
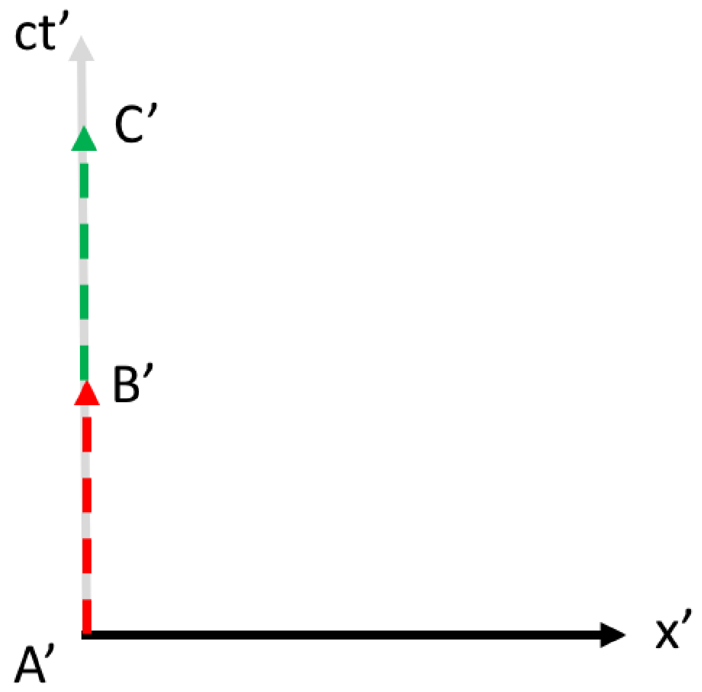
Let us **denote** (定義) :

Event A : The light signal is **emitted** (發射) from the **bottom mirror**.

Event B : The light signal **reaches** (到達) the **upper mirror**.

Event C : The light signal **returns** (回到) the **bottom mirror**.

We can hence draw 2 different space-time diagrams from the point of view of Doraemon and Yiu Yung.

Spacetime diagram as seen from Yiu Yung's Frame	Spacetime diagram as seen from Doraemon's Frame
	
Note : The light pulse has moved to the **right** as seen from Yiu Yung's frame.	Note : The light pulse is moving vertically up and down as seen from Doraemon's frame.

Example 5.1

Consider the case in Example 2.1 :

Background of the case :

At the instant shown, a Doraemon is standing in the middle of a moving bus travelling to the right at a speed u. Two of his friends, A and B, are standing at the 2 ends of the bus. Yiu Yung is at rest outside the bus.

At the time t = 0, Doraemon sends 2 light signals to A and B **simultaneously**. In his point of view, A and B will receive the light signals at the same time. But from Yiu Yung's point of view, A will receive the signal first.

(a) Sketch the spacetime diagram from the point of view of the middle Doraemon.

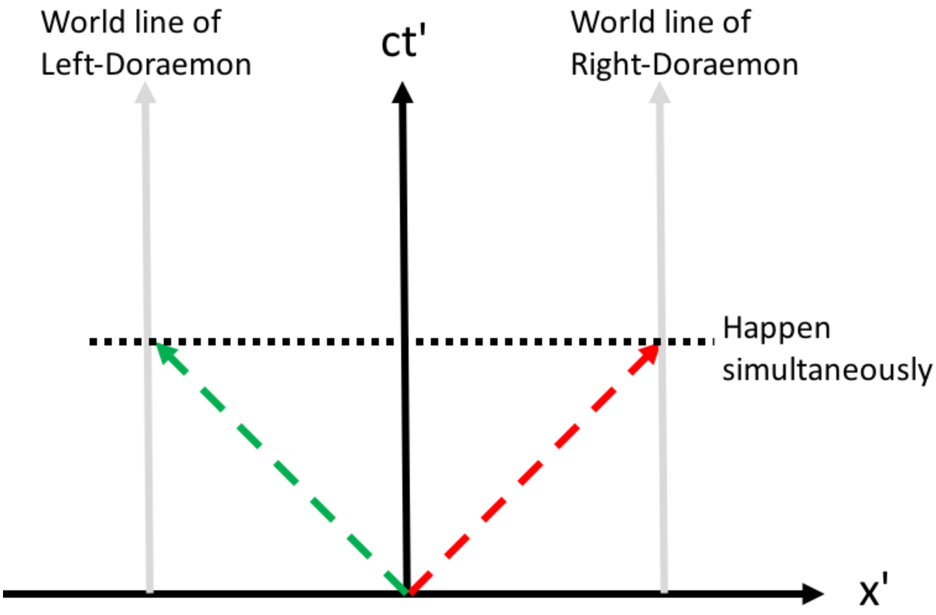
(b) Sketch the spacetime diagram from the point of view of Yiu Yung.

(c) What can be a possible conclusion to this case? How is it related to relativity?

[Solutions]

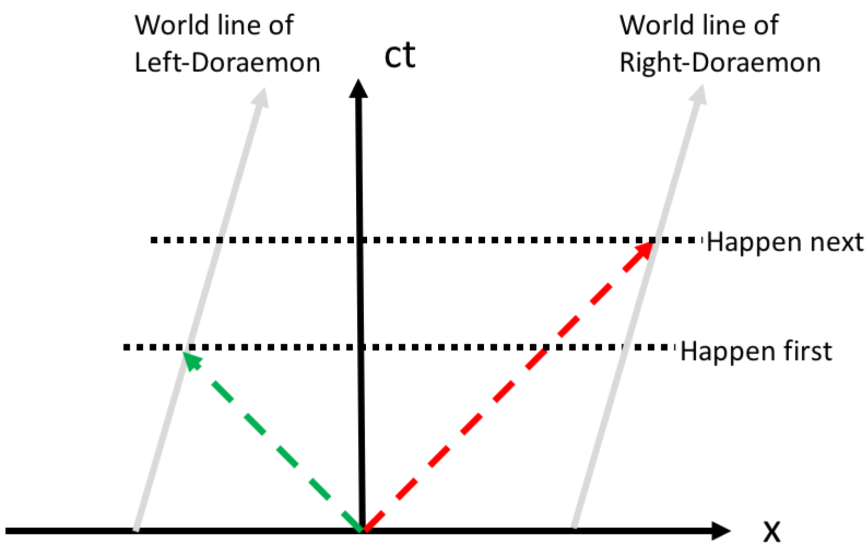
(a) Sketch the spacetime diagram from the point of view of the middle Doraemon.

[Sol]



(b) Sketch the spacetime diagram from the point of view of Yiu Yung.

[Sol]



(c) What can be a possible conclusion to this case? How is it related to relativity?

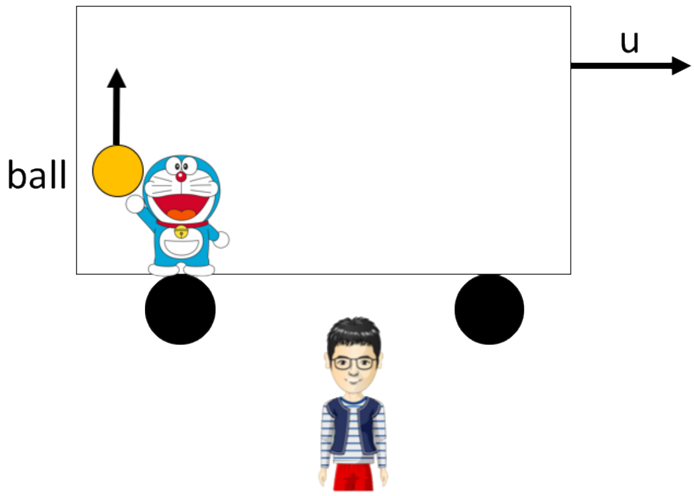
[Sol] From the point of view of Doraemon, Event A (light signal reaches A) and event B (light signal reaches B) happen at the same time (Simultaneously), but from the point of view of Yiu Yung, Event A happens before Event B.

Conclusion : Simultaneity is NOT an absolute concept in relativity.

(Or other acceptable answers)

Challenge 5.1

Consider the following case :



Doraemon is riding on a bus moving to the right with an uniform speed u. Yiu Yung is at rest outside the bus. At time t = 0, Doraemon throws a ball upward.

(a) Draw the spacetime diagram from the point of view of Doraemon.

(b) Draw the spacetime diagram from the point of view of Yiu Yung.

(c) Assume that both Doraemon and Yiu Yung have a proper clock to measure the time interval between the ball's motion. **State** who will measure the proper time.

(d) This time, Doraemon throws a ball to the **right** with a speed v, as measured from his frame. Find the speed of the ball as seen by Yiu Yung using Lorentz Transformation of velocity.

Spare some time and think a bit more…

- Is the ball itself a good **inertial reference frame**? Why?
- Consider the case in question (d). There is another Doraemon (Say B) at the right end of the bus to catch the ball. Compare the time elapsed between event A (Left Doraemon throws the ball) and event B (B catches the ball) from Doraemon's and Yiu Yung's point of view. Which one is longer? Can you explain why?

5.2 – Drawing on the Spacetime Diagram

In this section, we will **illustrate** (演示) how to draw and use the spacetime diagram.

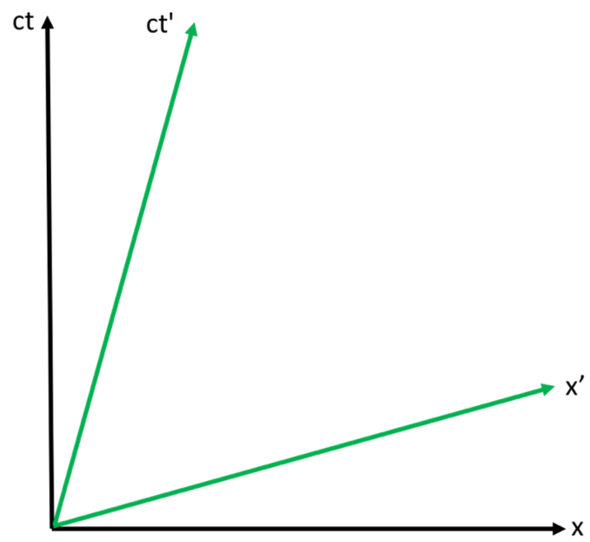
We now want to plot both S and S' frame on the same graph. First, we have to find the ct' and x' axes on the S frame.

Recall the Lorentz Transformation equations in Chapters 3 and 4 :

Equation 1	Equation 2
$x' = \gamma(x - ut)$	$x = \gamma(x' + ut')$

Equation 3	Equation 4
$t' = \gamma(t - \frac{ux}{c^2})$	$t = \gamma(t' + \frac{ux'}{c^2})$

What we are trying to do is to **merge** (合併) the 2 spacetime diagrams for S and S' frame on the graph paper. We try to put it in the form such that their origins (O and O') **coincide** (共點).



To make the 2 origins coincide, we require :
(a) x' = 0 when x = 0
(b) t' = 0 when t = 0

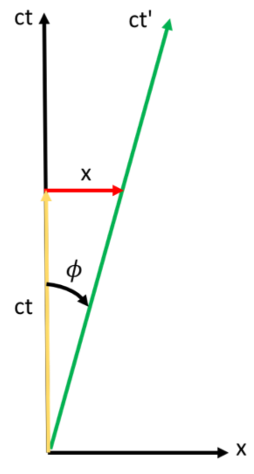
For requirement (a), and together with equation 2, we have :	$x = \gamma[(0) + ut'] = \gamma(ut')$
Recall that the **time dilation equation** in Chapter 2 is given by :	$t = \gamma t'$

From the 2 equations, we have

$$x = \gamma(ut') = \gamma[u(\frac{t}{\gamma})] = ut$$

as the equation for the ct' axis.
[Note : x is now measuring the distance of the S' frame with respect to its own origin.]

The slope of the line is given by u. So we have the angle ϕ between the ct axis and the ct' axis given by :



$$tan(\beta) = \frac{x}{ct}$$

$$\beta = tan^{-1}(\frac{x}{ct}) = tan^{-1}(\frac{ut}{ct}) = tan^{-1}(\frac{u}{c})$$

Similarly, for the x' axis,

For requirement (b), and together with equation 4, we have :	$t = \gamma[(0) + \frac{ux'}{c^2}] = \gamma\frac{ux'}{c^2}$
Recall that the **length contraction equation** in Chapter 2 is given by :	$x = \gamma x'$

From the 2 equations, we have

$$t = \gamma(\frac{ux'}{c^2}) = \gamma[\frac{u(\frac{x}{\gamma})}{c^2}]$$

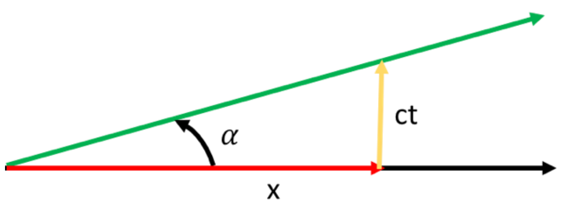
$$c^2 t = ux$$

as the equation for the x' axis.

The angle between the x' and x axis, α, is given by:

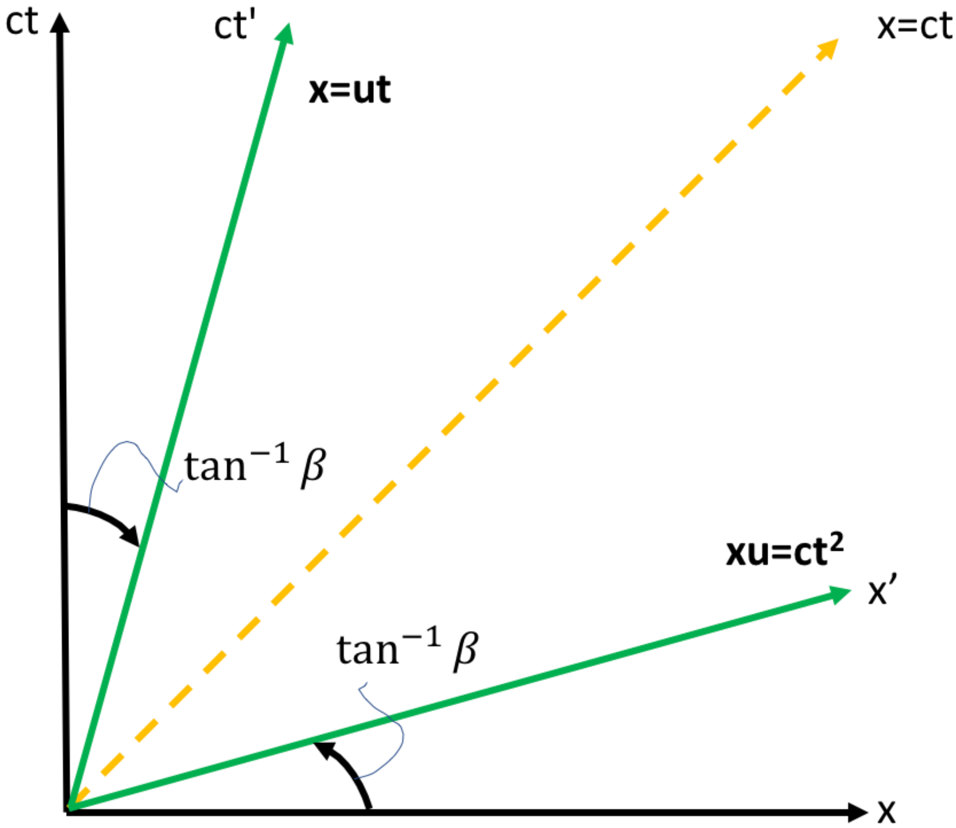
$$tan(\alpha) = \frac{ct}{x}$$

$$\alpha = tan^{-1}(\frac{ct}{x}) = tan^{-1}(\frac{\frac{ux}{c}}{x}) = tan^{-1}(\frac{u}{c})$$

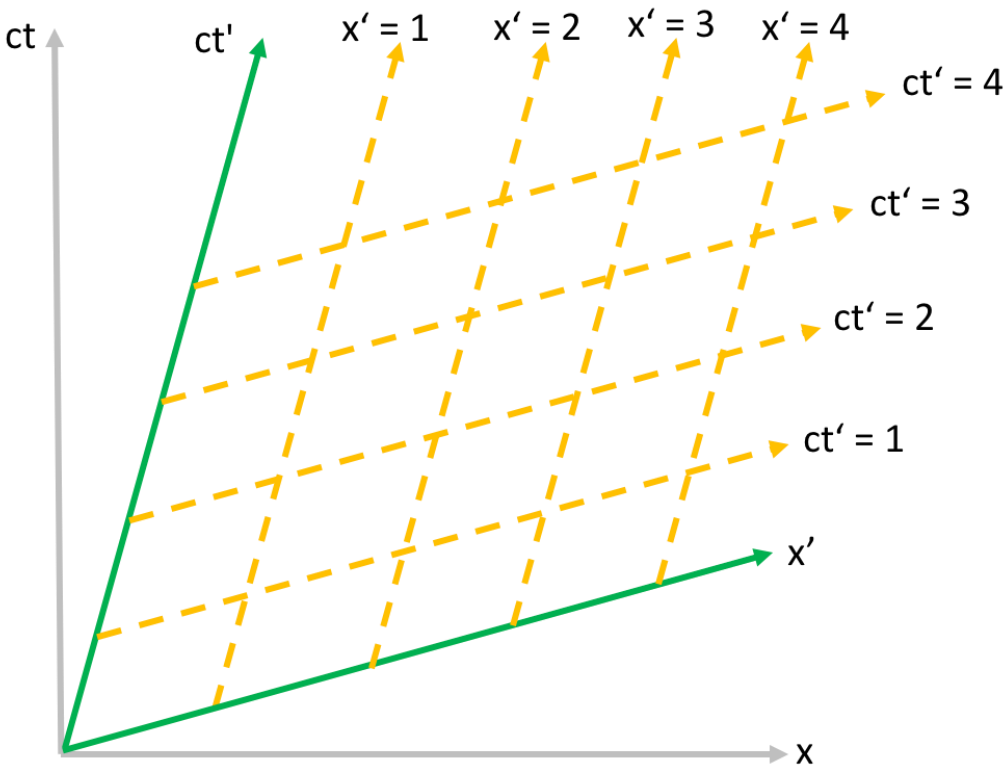


which is exactly the same as ϕ. We hence define the angle between the S and S' frame neighbouring axes as $\tan^{-1}\left(\frac{u}{c}\right) = \tan^{-1}\beta$.

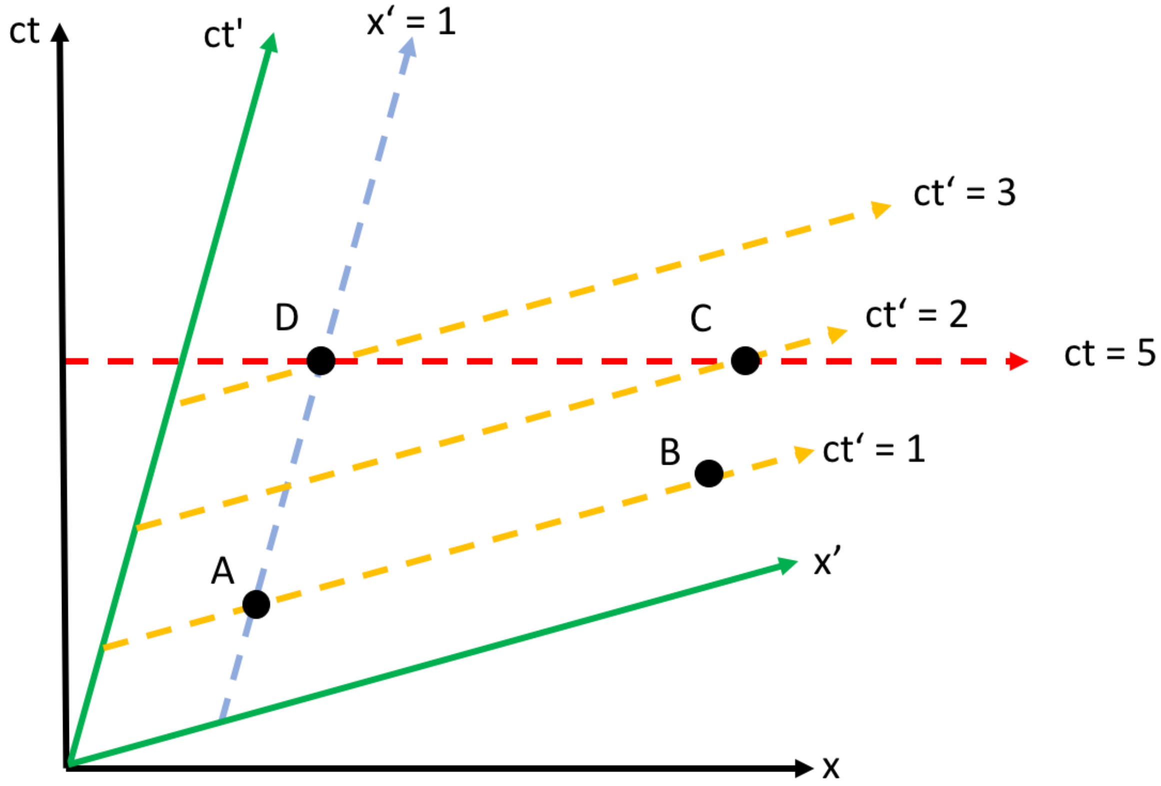
The spacetime diagram below summarizes the above results.



As the axis have tilted, the **grid lines** (格線) will also be tilted :



Let's consider 4 events A, B, C and D in the following spacetime diagram.



The yellow lines are called the "**lines of simultaneity**" (同步線) of the S' frame. Events lying on this line happen **simultaneously** in the frame S'. Similarly, the red line is the line of simultaneity of frame S.

We can easily see that while events A and B happen **simultaneously** in frame S', this is **NOT** the case in the S frame; Indeed, event A happens before B in the S frame.

On the other hand, while events C and D happen **simultaneously** in frame S, this is **NOT** the case in the S' frame. In fact, event C happens before D in the S' frame.

This once again prove that

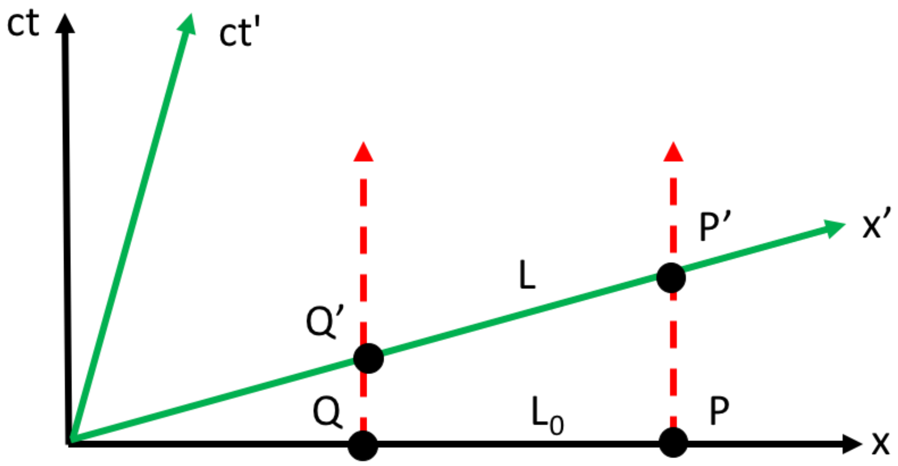
"**Simultaneity**" is NOT an absolute concept in relativity.

At the same time, the blue line is a **world line** (世界線) at x' = 1 in the S' frame. This tells us that event A and D happen at the **same location** in the S' frame, but this is **NOT** the case in the S frame! Actually, event D happens at the **right** of event A in the S frame.

We will see how we can make use of the spacetime diagram to prove length contraction and time dilation in **Example 5.2** and **Challenge 5.2**.

Example 5.2

Let's assume in the S-frame (rest frame), there is a rod of **proper length** L_0 at rest. Denote the left end and right end's position as Q and P respectively in the S frame. There is another moving S' frame with a speed u relative to the frame S. The following spacetime diagram illustrate the situation.



Prove the length contraction equation using the given materials and information.

[Solutions]

Let's denote the coordinates of Q, Q', P, P' as $Q(x_1, 0), Q'(x_1', 0), P(x_2, 0)$ *and* $P'(x_2', 0)$.

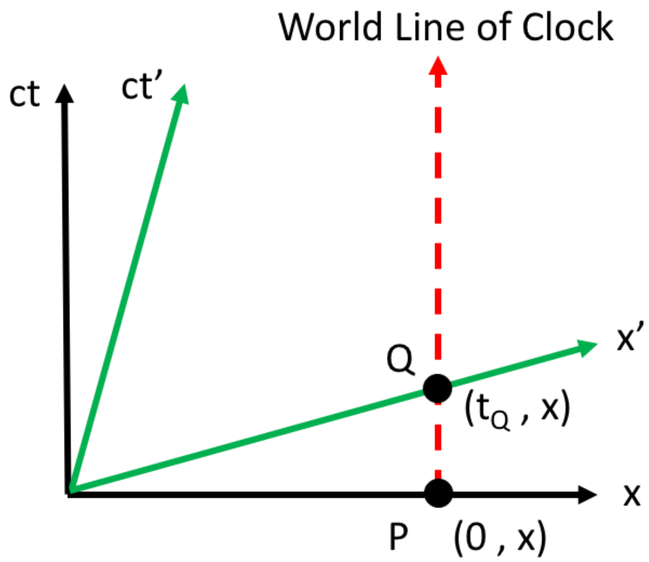
Note that $\Delta x' = \gamma(\Delta x - u\Delta t)$. We have:

$$x_2' - x_1' = \gamma[(x_2 - x_1) - u(0)] = \gamma(x_2 - x_1)$$
$$L_0 = \gamma L$$
$$L = \frac{L_0}{\gamma}$$

which is the length contraction equation.

Challenge 5.2

Let's assume in the S-frame (rest frame), at the position x and t = 0, there is a clock at rest. There is another moving S' frame with a speed u relative to the frame S. When the time of the clock reads t_q, the clock intersects with the x' axis of the S' frame. The following spacetime diagram illustrate the situation.



Derive the time dilation formula using the given materials and information.

Summary

Key Points

5.1 Spacetime Diagram

- There are usually 2 axes in the spacetime diagram :
  - Horizontal axis : Spatial position (x)
  - Vertical axis : Time axis (ct)

- A **world line** is the **trajectory** of any person or object on a spacetime diagram.

- The angle θ between the world line of any object and the vertical time axis can be related by the equation :

$$\tan\theta = \frac{\Delta x}{c\Delta t} = \frac{u}{c}$$

  where u is the speed of the object. Note that θ is always smaller than 90°.

5.2 Drawing on the Spacetime Diagram

- The equation of the ct' axis is given by :

$$x = ut$$

- The equation of the x' axis is given by :

$$ux = c^2 t$$

- The angle between the S and S' neighbouring axes is given by :

$$\tan^{-1}\beta = \tan^{-1}\left(\frac{u}{c}\right)$$

- A **line of simultaneity** is a line on the spacetime diagram on which all the events happen at the same time in that reference frame.

Key Terms

Astronomical unit	天文單位	P.1	Spacetime diagram	時空圖	P.3
Cartesian Coordinates	直角坐標系	P.3	Space axis	空間軸	P.3
Spatial position	空間上的位置	P.3	Event	事件	P.3
Time axis	時間軸	P.3	World line	世界線	P.5
Dimension	量綱	P.6	Spacetime interval	時空區間	P.7
Coincide	共點	P.13	Grid lines	格線	P.15
Line of simultaneity	同步線	P.16			

Check Your Concepts

1. Can you clearly define what a **world line** is? [**Section 5.1**]

2. What is the angle between the world lines of an at rest observer and a moving observer? [**Section 5.1**]

3. What is a "line of simultaneity"? [**Section 5.2**]

Historical Profile

Bernhard Riemann was a German mathematician who made contributions to analysis, number theory, and differential geometry. In the field of real analysis, he is mostly known for the first rigorous formulation of the integral, the Riemann integral, and his work on Fourier series. Through his pioneering contributions to differential geometry, Bernhard Riemann laid the foundations of the mathematics of general relativity.

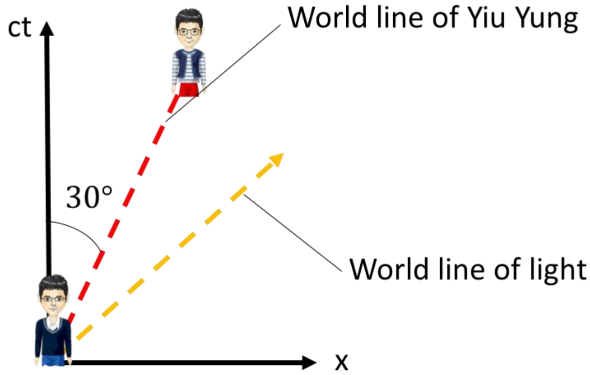
Chapter Exercise

Multiple Choice Questions

1. Consider the spacetime diagram for an inertial reference frame. How will the world line look like if it is at rest at some position x = a from your point of view? The line will be...
   A. Oblique
   B. Horizontal
   C. Vertical
   D. Not enough information is given to deduce the answer.

2. Which of the following best shows the equation of the world line for light in an inertial reference frame?
   A. $x = t$
   B. $ux = c^2t$
   C. $x = ut$
   D. $x = ct$

3. Refer to the following spacetime diagram. Alvin is at rest (S frame) while Yiu Yung (red line) is moving at a speed u away from Alvin.



   Determine the value of Yiu Yung's speed u using the information given.
   A. $0.5c$
   B. $\frac{\sqrt{3}}{3}c$
   C. $2c$
   D. Not enough information

Question 4 – 5 refer to the following.

Event A and B lie on the same horizontal line in a spacetime diagram of an at rest observer.

4. What is the name given to the horizontal line mentioned?
   A. World line.
   B. Time line.
   C. Line of simultaneity.
   D. Yellow line.

5. Assumes that the observers live in a 1D world (they can only move left or right). If another observer C sees event A happens before event B, and given that event A's position is on the left of event B in the rest observer's point of view. To which direction is C moving towards?
   A. To the left of the rest observer.
   B. To the right of the rest observer.
   C. C is also at rest.
   D. C first moves to the left, then back to the right again.

Short Questions

1. By making use of a spacetime diagram, show that observers moving relative to each other can have different opinions on the simultaneity of two events A and B.

2. Show explicitly that the world line in a spacetime diagram is described by the equation $x = ct$. You should list out all the mathematical steps required to achieve the result.

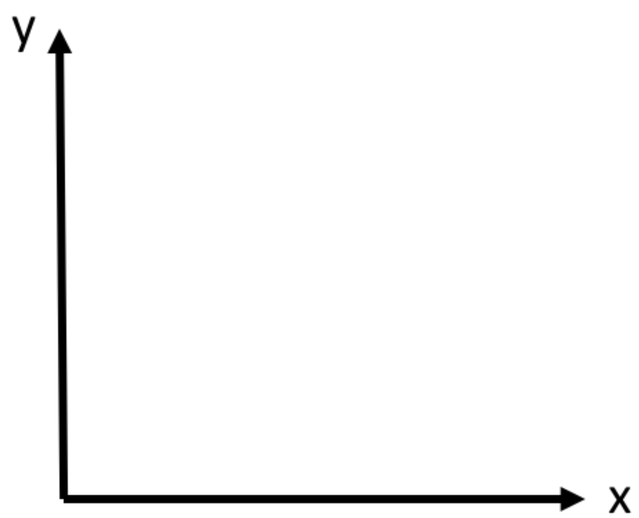
Structured Questions

[Question 1] (Difficulty : ⋆)

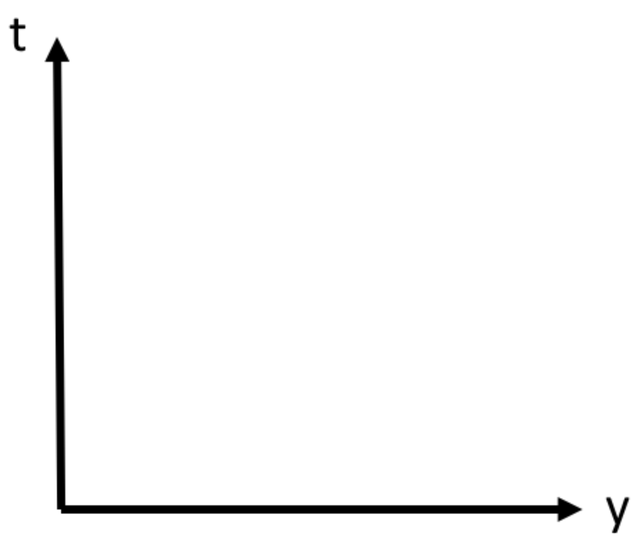
We have been dealing with 1D problems only so far in this chapter (i.e. We only consider the x-direction as the only spatial coordinates). Let us consider one more spatial coordinates such that the spacetime coordinates of each event is (t, x, y).

Let's consider a man standing at $(x, y) = (0,0)$. At time t = 0, he throws a ball upward towards the positive y-direction. Assume gravity acts along the negative y-direction.

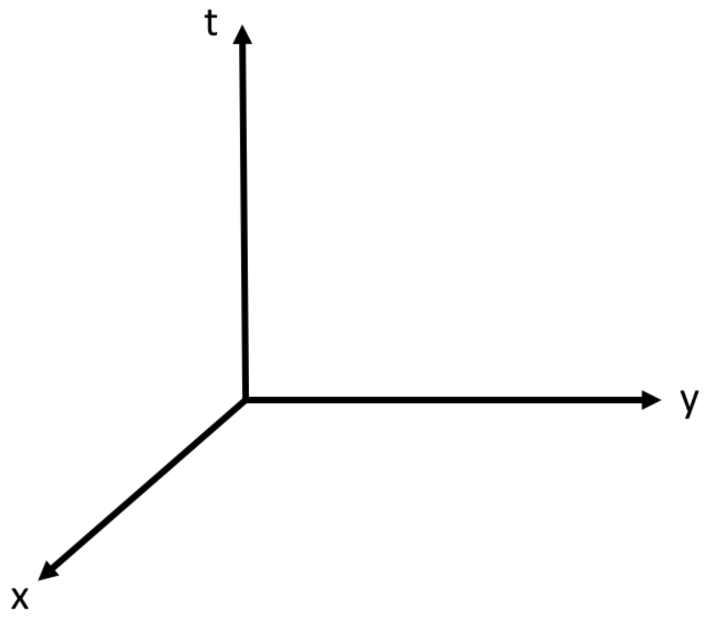
(a) Sketch, on the x-y plane below, the trajectory of the ball when the ball is thrown.



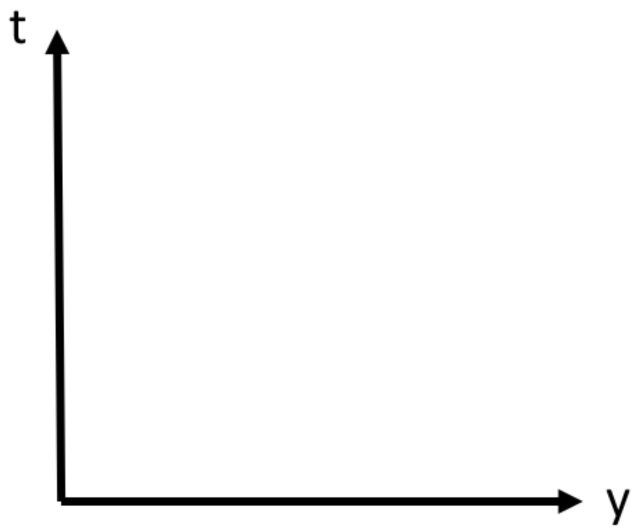
(b) Sketch, on the y-t plane below, the trajectory of the ball when the ball is thrown.



(c) Sketch, on the x-y-t system below, the trajectory of the ball when the ball is thrown.



(d) Sketch, on the y-t plane below, the trajectory of the ball if there is NO GRAVITY.



[Question 2] (Difficulty : ⋆ ⋆)

Consider the case below.

Background of the case : Inside a moving car, there are **2 mirrors**. At time t = 0, a light signal is sent from the **bottom mirror** to the **upper mirror**. After being **reflected** from the **upper mirror**, it returns to the bottom mirror. When the light signal is first sent, both **Doraemon** (on the car) and **Yiu Yung** (on the road at rest) both start a timer to count the time of travel of the light signal.

(a) Sketch a spacetime diagram showing both the S-frame (Yiu-Yung's frame) and the S'-frame (Doraemon's frame).

Denote :

- Event A = Light emitted from the base.
- Event B = Light returned to the base.

Let the coordinates of A' = $A'(0,0)$ and B' = $B'(0,t_2')$ in the S' frame.

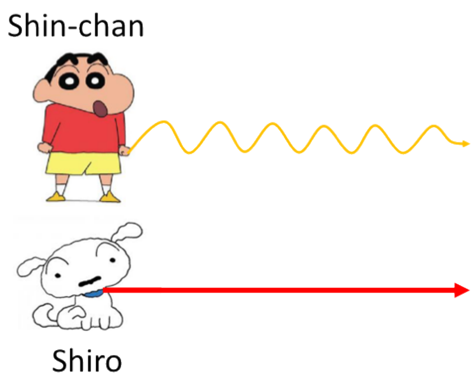
(b) Prove the time dilation equation using the information above.

(c) Prove the length contraction equation using the information above.

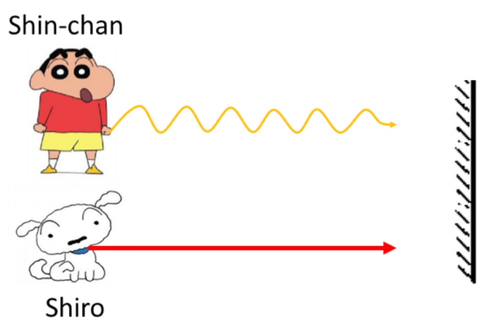
(d) If, after event B, the car suddenly move backward with speed u and a light pulse is emitted from the base again immediately. Ignore the acceleration involved in this process. Denote Event C = 2^{nd} Light pulse emitted from the base, and Event D = 2^{nd} light pulse returned to the base. Sketch the new situation on the same spacetime diagram in (a).

[Question 3] (Difficulty : ★ ★ ★)

In a Japanese cartoon series **"Crayon Shin-chan"** (蠟筆小新), the main character Shin-chan (小新) has a pet dog named Shiro (小白). One day, Shin-chan plays a game with Shiro. He stands at rest at a position together with Shiro. At time t = 0, he sends a light signal to the right and Shiro immediately follows the signal at a speed of u. (You may assume that u is a fraction of c, the speed of light.)



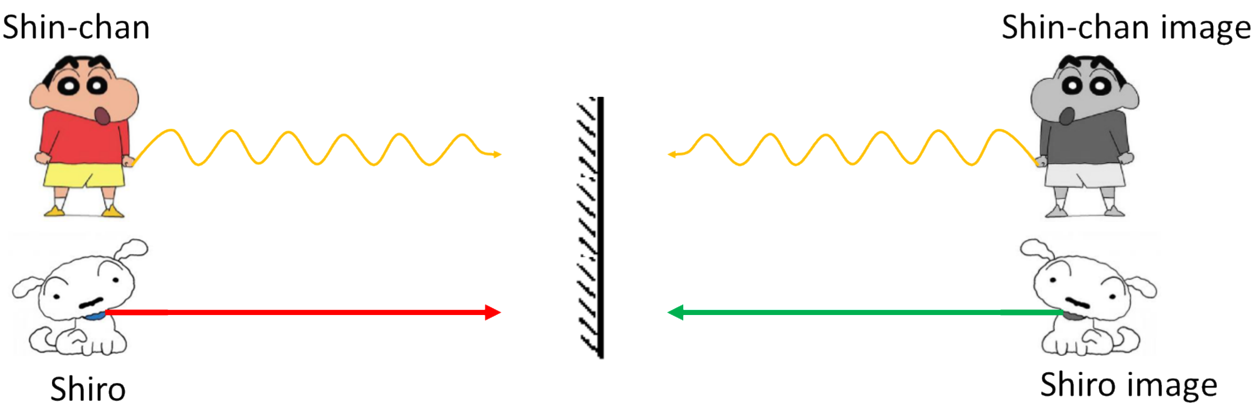
(a) Sketch a spacetime diagram showing the world lines of Shin-Chan, Shiro and the light signal.



(b) There is a mirror at a distance **x = a** from Shin-Chan. Use **dotted line** to represent the world line of the mirror in the same spacetime diagram in (a).

(c) The light is reflected and returns back to Shin-Chan after hitting the mirror. Label the point **R** as the point when the light is reflected, and **T** as the point when Shiro catches up with the reflected light ray.

In fact, there will be an image of Shin-Chan and Shiro behind the mirror during the whole process.



(d) Sketch the world line of Shiro image, Shin-Chan image and the light ray image on the same spacetime diagram in (c). Denote **T'** as the point when Shiro image catches up with the reflected image light ray, and R' as the point when the image light ray is reflected. Hence show that R and R' coincide.

(e) From Shiro's point of view, what is the speed of Shiro's image? (Hint : Use Lorentz's transformation.)

(f) From Shiro image's point of view, what is the speed of Shiro? (Hint : Use Lorentz's transformation.)

(g) Using your spacetime diagram, show that

(1) From Shin-Chan's point of view, T and T' happen simultaneously.

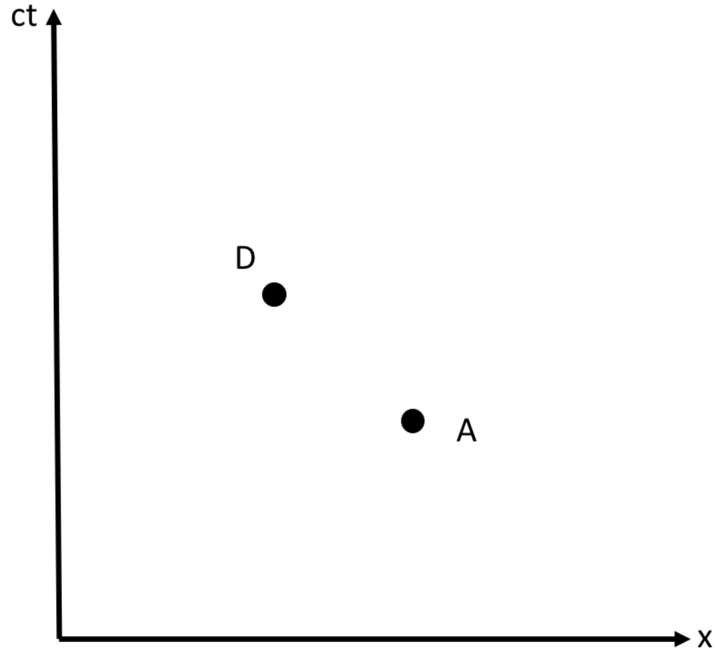
(2) From Shiro's point of view, T' happens before T.

(3) From Shiro image's point of view, T happens before T'.

(h) What conclusion can you make from the above results? How is it related to relativity?

[Question 4] (Difficulty : ★ ★ ★ ★)

Consider the following spacetime diagram.



A and D represents 2 events happening in the rest frame.

(a) Does event A happen before D? Or does D happen before A?

(b) Does A and D happen at the same spatial position in the rest frame?

(c) Suggest a way for another observer K to move such that he will see that A and D happen **at the same place**. Verify your answer by drawing the world line ct' and the spatial axis x' of the observer, as well as the world line of event A and D in his frame.

(d) Suggest a way for another observer L to move such that he will see that A and D happen **at the same time**. Verify your answer by drawing the world line ct'' and the spatial axis x'' of the observer, as well as the **line of simultaneity** for the events A and D in his frame.

(e) If we want to see event D happens before event A, how fast should we move? Use a spacetime diagram to help you. (Note : Here you will see the consequences of moving faster than the speed of light. Suppose event A marks your birth, and event D marks the 1st day you go to school, then in this case you will go to school even before you were born…)

[Question 5] (Difficulty : ★ ★ ★ ★ ★)

In 2200, the Earth has developed spaceships which can fly at speed very close to the speed of light. In a certain year, NASA sends 2 spaceships outward to look for aliens. **Spaceship 1** flies towards the positive x-direction at a speed of 0.2 c, while **Spaceship 2** flies towards the negative x-direction at a speed of 0.4 c. Assume that t = 0 when the spaceships departs, and assume that NASA space station is at rest at x = 0.

(a) Sketch the world lines of the NASA space station, spaceship 1 and 2 on the same spacetime diagram.

(b) At t = 2, NASA received a warning from an unknown alien, and it immediately issue a warning signal to both **spaceship 1 and 2**. Sketch the world lines of the light warning signals.

(c) According to your diagram in (b), which spaceship will receive the warning signal first? Where is the other spaceship relative to NASA space station when the 1st spaceship receives the signal?

(d) When the spaceships receive the signal, they will return to the NASA space station at a speed of 0.5 c. Sketch the world line of the spaceship which 1st receive the warning signal.

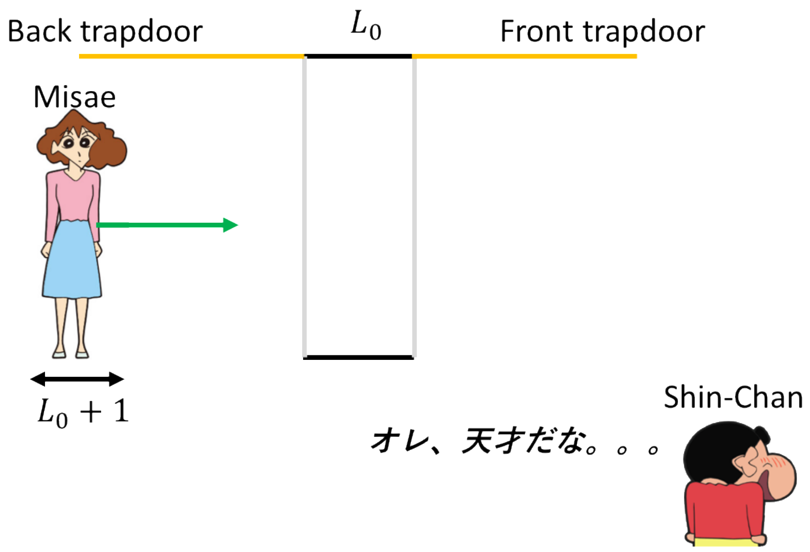
(e) When the 1st spaceship receives the signal, the other spaceship, unfortunately, relative to the NASA space station, is simultaneously captured by the evil alien and it rides the spaceship back to the NASA space station. What should be the speed of the captured spaceship such that it can return to the NASA space station **at the same time** as the other spaceship does?

(f) The alien rides the spaceship at the speed in (e). However, unfortunately for the alien but fortunately for the Earth, the spaceship exploded **half-way** along the path it returns to NASA space station. Denote the event of the explosion as V. Sketch the world lines of the light emitted at the explosion. Will the light signal reaches NASA space station first, or will the other spaceship returns first?

(g) The astronaut together with the captured spaceship remains at rest at his position after the explosion of the spaceship. After receiving the explosion light signal, NASA immediately sends a rescue spaceship travelling at a speed of 0.8 c to save the astronaut. By how much time after the explosion will the astronaut be saved?

[Question 6] (Difficulty : ★ ★ ★ ★ ★ ★)

Misae and trap paradox



In a Japanese cartoon series **"Crayon Shin-chan"** (蠟筆小新), the main character Shin-chan (小新) always describes his mother Misae (美冴) as a "big fat old witch". After learning special relativity today at school, he thinks of a "great idea" to make fun of his mother.

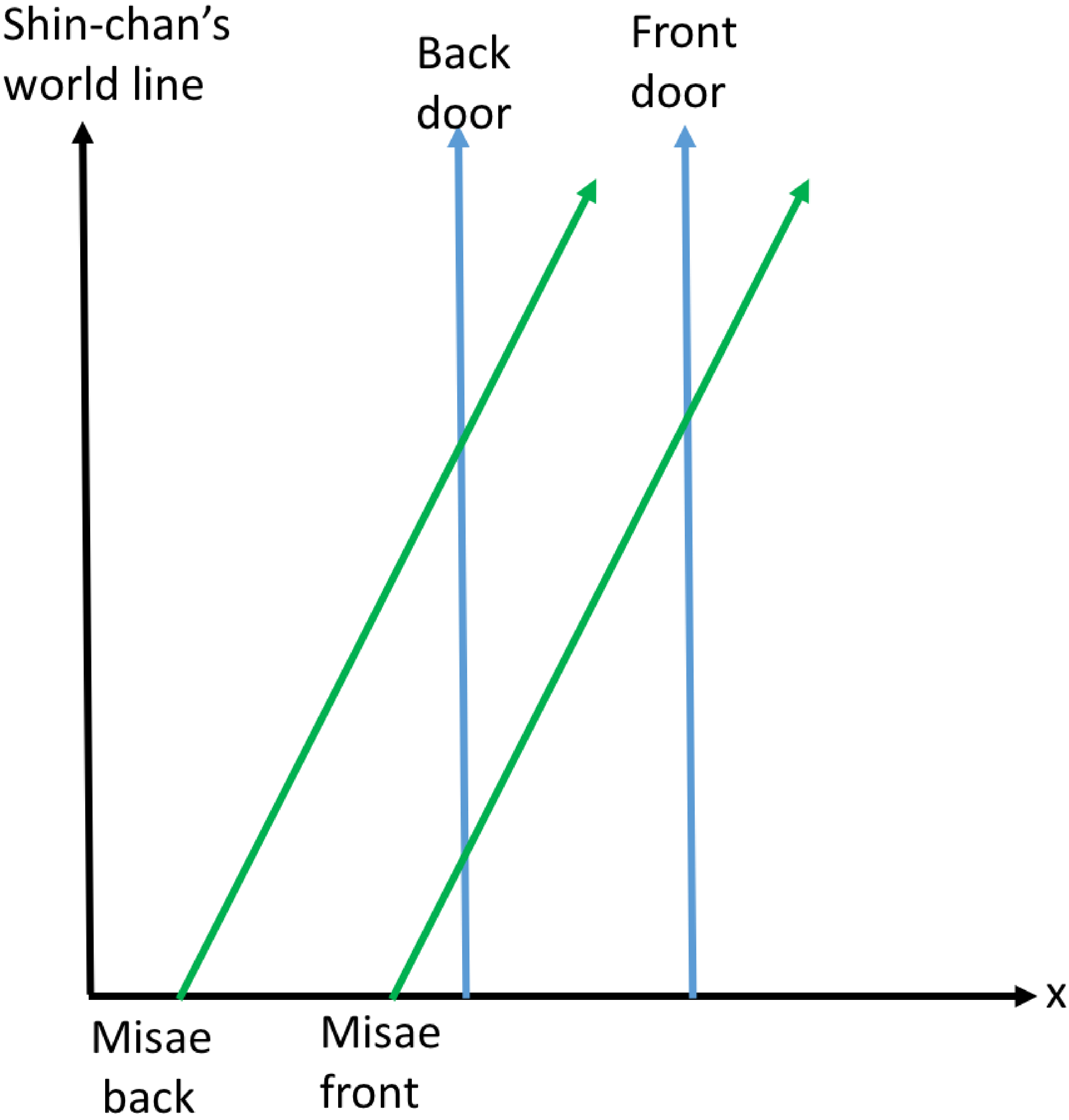
Noticing that his mother's horizontal proper length is $L_0 + 1$, Shin-Chan designed a trap of length L_0. He makes Misae angry such that she chases him at a speed of u (fraction of c) which is fast enough, according to the length contraction formula, to contract Misae's length lower that L_0 such that she can fit into the trap.

Once Misae enters completely into the trap, Shin-Chan will close both the front and back trap door simultaneously to trap Misae inside. From the point of view of Shin-Chan, this is completely possible.

However, from the point of view of Misae, it is the "trap" which is moving, and the trap would indeed undergo contraction and will be too small to trap her.

What is going on here? Who is right and who is wrong? Can you figure out what is the thing that confuse you here?

Try to sketch a spacetime diagram to help you resolve this paradox. You may find part of the completed spacetime diagram in the next page useful.


Shin-chan's world line
Back door
Front door
x
Misae back
Misae front

~ The End~

Chapter 6 – More About Spacetime Diagrams

Chapter Starters…

Readers, I understand that you must be a bit angry because you cannot actually do ANY DIRECT MEASUREMENTS on spacetime diagrams in the last chapter. In fact I can sense that some of you are saying that I have DECEIVED (欺騙) you. I'm sorry, but that what I meant to do to you… BAZINGA (Credit to Sheldon Cooper from "The Big Bang Theory"). A good news to you is, in this chapter, we will be able to formulate an ACTUALLY APPLICABLE spacetime graph such that we can do real measurements on it. Are you happy to know that? I bet you DID. But before so, let us give you a short review on COORDINATE GEOMETRY (座標幾何).

In figure A, 2 points $P(x_1, y_1)$ *and* $Q(x_2, y_2)$ are shown on the x-y coordinate plane.

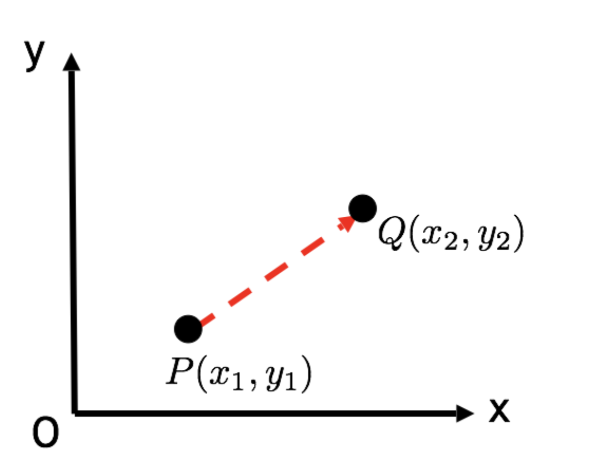


Figure A

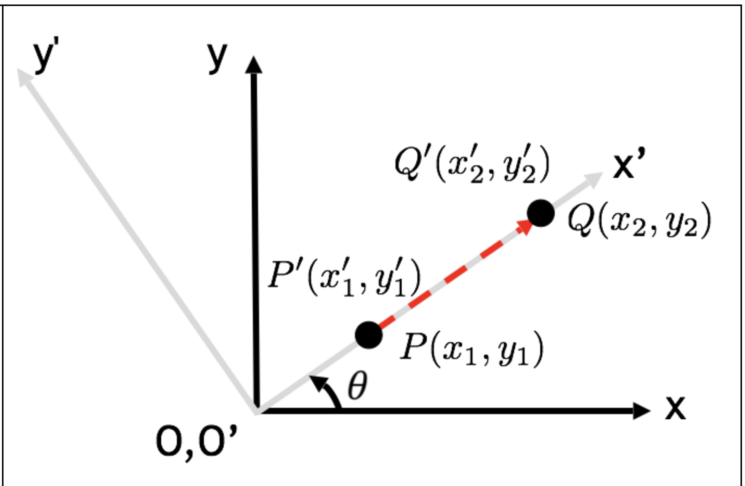


Figure B

(a) Express the distance between P and Q (i.e. PQ), in terms of x_1, y_1, x_2 *and* y_2.

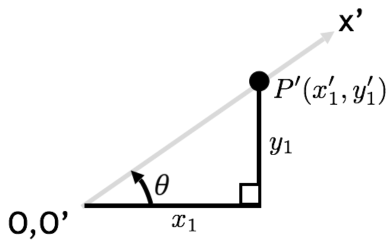
(b) In figure B, the x and y axes are rotated (旋轉了) anti-clockwisely (逆時針地) by an angle θ such that we get a **new coordinate system with axes** x'-y'. The coordinates for points P and Q are $P'(x_1', y_1')$ and $Q'(x_2', y_2')$ respectively in the new system, and P and Q lie on the x'-axis.

(i) Express x_1' and y_1' in terms of x_1 and y_1.

(Hints :

(1) For x_1', consider the triangle on the right :

(2) For y_1', note that P' lies **on the x' axis**.)



(ii) Similarly, express x_2' and y_2' in terms of x_2 and y_2.

This Question continues next page.

(iii) Express the distance between P' and Q' (i.e. P'Q') in terms of x_1' and x_2'. Using your answer in (ii), further express your answer in terms of x_1, y_1, x_2 and y_2.

(Notes : You might be curious that P'Q' is **NOT OBVIOUSLY EQUAL** to your answer in (a). But you do know that it **MUST BE** equal. What's wrong?)

(c) Express $\tan\theta$ in terms of

(i) x_1 and y_1.

(ii) x_2 and y_2

(d) Using (c), show that your answer in (a) can be expressed as

$$PQ = (x_2 - x_1)\sqrt{1 + \tan^2\theta}$$

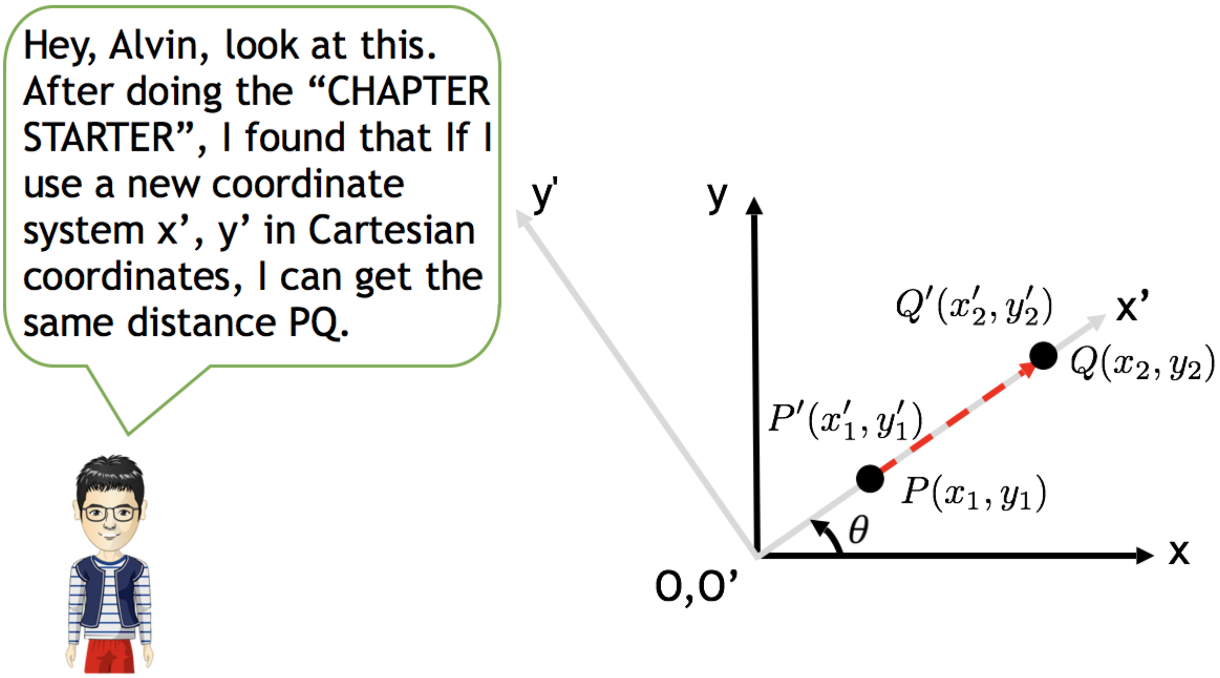
(e) Using (c), show that your answer in (b)(iii) can be expressed as

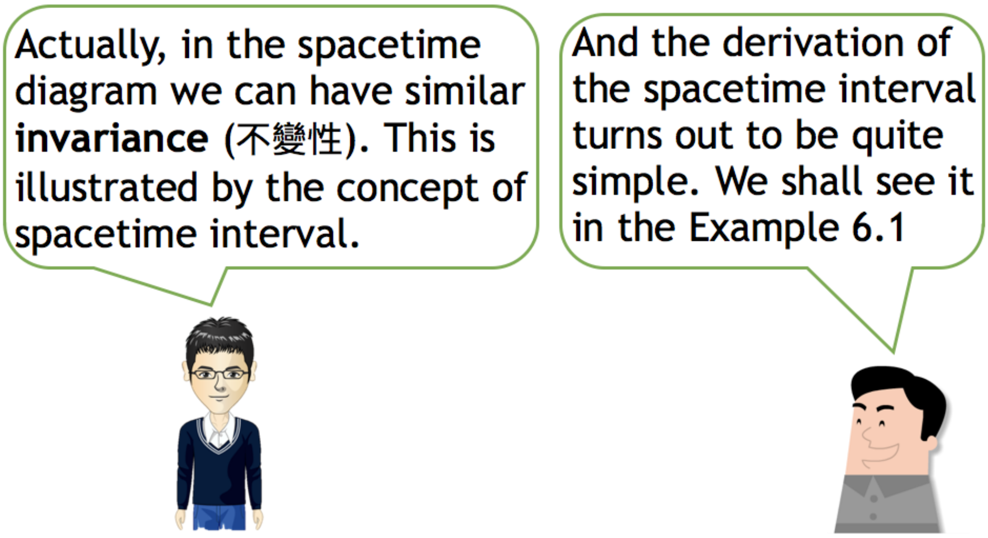
$$P'Q' = (x_2 - x_1)\sqrt{1 + \tan^2\theta}$$

Notes : You should notice that your answers in (d) and (e) are the same. This shows that the LENGTH between 2 points on a coordinate plane is **INVARIANT** (不變的) under coordinate frame TRANSFORMATION (轉換). In the beginning of this Chapter, we will focus on figuring out a **similar quantity** like **LENGTH** which is also invariant under transformation of the **spacetime axes**.

6.1 – Spacetime Interval

In this section we will introduce the idea of spacetime interval (as similar to distance between 2 points in the x-y coordinate plane).



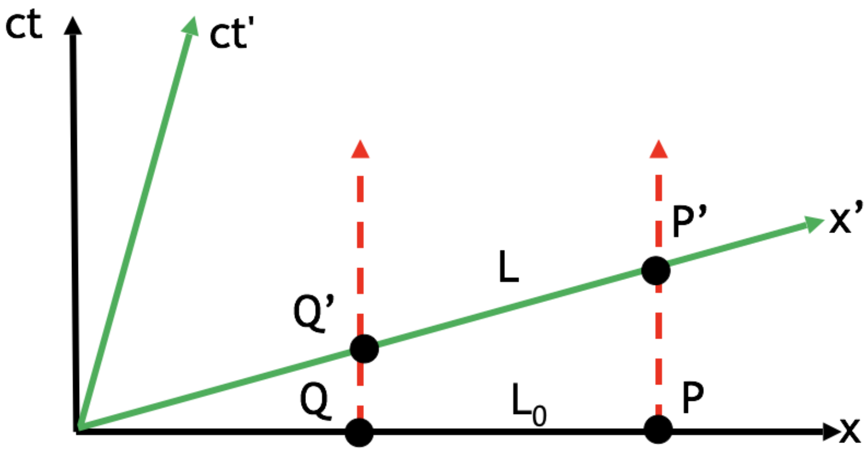


Spacetime interval (時空區間) is like the "**distance**" between 2 points in a **coordinate plane**, but in the **context** (背景) of relativity, this "distance" refers to the "**Spacetime distance**" between 2 **events** (事件) in a spacetime diagram. We want this **physical quantity** (物理量) [i.e. **Spacetime interval**] to be **invariant** (不變的) [That will not be changed under frame **transformation** (轉換)] as it is like for **length** between 2 points on the **coordinate plane**.

We will show you that the expression foe **spacetime interval** is somehow similar to that of **Pythagoras Theorem** (畢達哥拉斯定理), but with a slight difference.

Example 6.1

Let's assume in the S-frame (which is an inertial rest frame), there is a rod of **proper length** $\mathbf{L_0}$ at rest. Relative to the S-frame, there is another moving inertial reference frame S'. At t = 0, the back end and the front end of the rod is at point Q and P on the x-axis in the S-frame respectively. After some time, the front end and the back end of the rod intersects the x'-axis at point Q' and P' respectively. The following spacetime diagram illustrates the situation.



(a) Which **TWO events** (out of events P, P', Q, Q') happen **simultaneously** in the S-frame? How about in the S'-frame? Are your answers the same for both frame? How is this related to an important concept in **relativity**?

(b) Denote $\mathrm{Q}'(0, \mathrm{x}'_\mathrm{Q})$ and $\mathrm{P}'(0, \mathrm{x}'_\mathrm{P})$ as the coordinates (t', x') for Q' and P' in the S' frame.

Compute the spatial difference $\Delta\mathrm{x}' = x'_2 - x'_1$ and the time difference $\mathrm{c}\Delta\mathrm{t}' = c(t'_2 - t'_1)$ between events Q' and P' in the S' frame in terms of x'_Q and x'_P.

(c) Denote $\mathrm{Q}'(\mathrm{t_q}, \mathrm{x_q})$ and $\mathrm{P}'(\mathrm{t_p}, \mathrm{x_p})$ as the coordinates (t, x) for Q' and P' (**NOT** Q and P) in the S frame. Compute the spatial difference $\Delta\mathrm{x} = x_2 - x_1$ and the time difference $\mathrm{c}\Delta\mathrm{t} = c(t_2 - t_1)$ between events Q' and P' in the S frame in terms of x'_Q and x'_P.

(Hint : You need to use **Lorentz Transformation**.)

(d) Let's define a quantity called the "**fake**" (虛假的) spacetime interval

$$\Delta\bar{\mathrm{S}}^2 = (c\Delta t)^2 + (\Delta x)^2$$

Compute $\Delta\bar{\mathrm{S}}^2$ (in S frame) and $\Delta\bar{\mathrm{S}}'^2$ (in S' frame) for events Q' and P'. Hence show that $\Delta\bar{\mathrm{S}}^2 \neq \Delta\bar{\mathrm{S}}'^2$.

(e) Show that if $\Delta\mathrm{S}^2 = -(c\Delta t)^2 + (\Delta x)^2$, then $\Delta\mathrm{S}^2 = \Delta\mathrm{S}'^2$.

[Solutions]

(a) Which **TWO events** (out of events P, P', Q, Q') happen **simultaneously** in the S-frame? How about in the S'-frame? Are your answers the same for both frame? How is this related to an important concept in **relativity**?

[Sol]

In the S-frame : Events P and Q

In the S'-frame : Events P' and Q'

The answers are **NOT** the same. This again shows that "**Simultaneity** is not an absolute idea in relativity.)

(b) Denote $\mathrm{Q}'(0,\mathrm{x}'_{\mathrm{Q}})$ and $\mathrm{P}'(0,\mathrm{x}'_{\mathrm{P}})$ as the coordinates (t', x') for Q' and P' in the S' frame.

Compute the spatial difference $\Delta\mathrm{x}' = x'_2 - x'_1$ and the time difference $\mathrm{c}\Delta\mathrm{t}' = c(t'_2 - t'_1)$ between events Q' and P' in the S' frame in terms of x'_{Q} and x'_{P}.

[Sol]

$$\Delta\mathrm{x}' = x'_P - x'_Q$$

$$c\Delta t' = 0$$

(c) Denote $\mathrm{Q}'(\mathrm{t}_{\mathrm{q}},\mathrm{x}_{\mathrm{q}})$ and $\mathrm{P}'(\mathrm{t}_{\mathrm{p}},\mathrm{x}_{\mathrm{p}})$ as the coordinates (t, x) for Q' and P' (**NOT** Q and P) in the S frame. Compute the spatial difference $\Delta\mathrm{x} = x_2 - x_1$ and the time difference $\mathrm{c}\Delta\mathrm{t} = c(t_2 - t_1)$ between events Q' and P' in the S frame in terms of x'_{Q} and x'_{P}.

(Hint : You need to use **Lorentz Transformation**.)

[Sol]

$$\Delta\mathrm{x} = x_p - x_q = \gamma(x'_P + ut'_P) - \gamma(x'_Q - ut'_Q) = \gamma(x'_P + u(0)) - \gamma\left(x'_Q - u(0)\right)$$

$$= \gamma(\mathrm{x}'_{\mathrm{P}} - x'_Q)$$

$$c\Delta t = c(\mathrm{t}_{\mathrm{p}} - t_q) = c\left[\gamma\left(t'_P + \frac{ux'_P}{c^2}\right) - \gamma\left(t'_Q + \frac{ux'_Q}{c^2}\right)\right] = c\left[\gamma\left(0 + \frac{ux'_P}{c^2}\right) - \gamma\left(0 + \frac{ux'_Q}{c^2}\right)\right]$$

$$= \frac{\gamma u}{c}(x'_P - x'_Q)$$

(d) Let's define a quantity called the "**fake**" (虛假的) spacetime interval

$$\Delta\bar{S}^2 = (c\Delta t)^2 + (\Delta x)^2$$

Compute $\Delta\bar{S}^2$ (in S frame) and $\Delta\bar{S}'^2$ (in S' frame) for events Q' and P'. Hence show that $\Delta\bar{S}^2 \neq \Delta\bar{S}'^2$.

[Sol]

$$\Delta\bar{S} = (c\Delta t)^2 + (\Delta x)^2 = \left[\frac{\gamma u}{c}(x'_P - x'_Q)\right]^2 + [\gamma(x'_P - x'_Q)]^2 = \gamma^2(x'_P - x'_Q)^2\left(1 + \frac{u^2}{c^2}\right)$$

$$\Delta\bar{S}' = (c\Delta t')^2 + (\Delta x')^2 = (0)^2 + (x'_P - x'_Q)^2 = (x'_P - x'_Q)^2$$

Obviously, $\Delta\bar{S}^2 \neq \Delta\bar{S}'^2$. (Notes : **Pythagoras Theorem** doesn't seem to work in relativity…)

(e) Show that if $\Delta S^2 = -(c\Delta t)^2 + (\Delta x)^2$, then $\Delta S^2 = \Delta S'^2$.

[Sol]

$$\Delta\bar{S} = -(c\Delta t)^2 + (\Delta x)^2 = -\left[\frac{\gamma u}{c}(x'_P - x'_Q)\right]^2 + [\gamma(x'_P - x'_Q)]^2 = \gamma^2(x'_P - x'_Q)^2\left(1 - \frac{u^2}{c^2}\right)$$

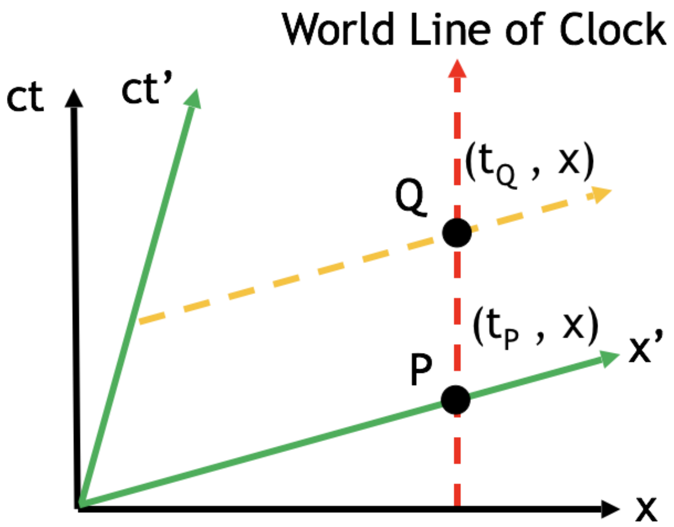
$$= \frac{\gamma^2(x'_P - x'_Q)^2}{\gamma^2} = (x'_P - x'_Q)^2$$

$$\Delta\bar{S}' = -(c\Delta t')^2 + (\Delta x')^2 = -(0)^2 + (x'_P - x'_Q)^2 = (x'_P - x'_Q)^2$$

Obviously, $\Delta\bar{S}^2 = \Delta\bar{S}'^2$. (Notes : In relativity, **Pythagoras Theorem** is still the same, only there is an **additional negative sign** in front of the "time difference".)

Challenge 6.1

Let's assume in the rest inertial reference frame (S-frame), there is a clock at rest at the position $x = x$. There is another inertial reference frame (S'-frame) moving at a speed u relative to the S-frame. At $t = t_P$, the clock intersects the x'-axis of the S'-frame. The point Q indicates the space-time position of the clock when $t = t_Q$. The following spacetime diagram illustrates the situation.



(a) Event P and Q happen at the same place in the S-frame. How about in the S'-frame?

(b) Compute the spatial difference $\Delta x = x_2 - x_1$ and the time difference $c\Delta t = c(t_2 - t_1)$ between events Q' and P' in the S-frame in terms of t_p and t_Q.

(c) Denote $Q'(t'_Q, x'_Q)$ and $P'(t'_P, x'_p)$ as the coordinates (t, x) for Q' and P' (**NOT** Q and P) in the S'-frame. Compute the spatial difference $\Delta x' = x'_2 - x'_1$ and the time difference $c\Delta t' = c(t'_2 - t'_1)$ between events Q' and P' in the S'-frame in terms of t_P and t_Q.

(Hint : You need to use **Lorentz Transformation**.)

(d) Let's define a quantity called the "**fake**" (虛假的) spacetime interval

$$\Delta\bar{S}^2 = (c\Delta t)^2 + (\Delta x)^2$$

Compute $\Delta\bar{S}^2$ (in S-frame) and $\Delta\bar{S}'^2$ (in S'-frame) for events Q' and P'. Hence show that $\Delta\bar{S}^2 \neq \Delta\bar{S}'^2$.

(e) Show that if $\Delta S^2 = -(c\Delta t)^2 + (\Delta x)^2$, then $\Delta S^2 = \Delta S'^2$.

From Example 6.1 and Challenge 6.1, we can see that **spacetime interval** (時空區間) ΔS^2 is defined by

$$\Delta S^2 = -(c\Delta t)^2 + \Delta x^2$$

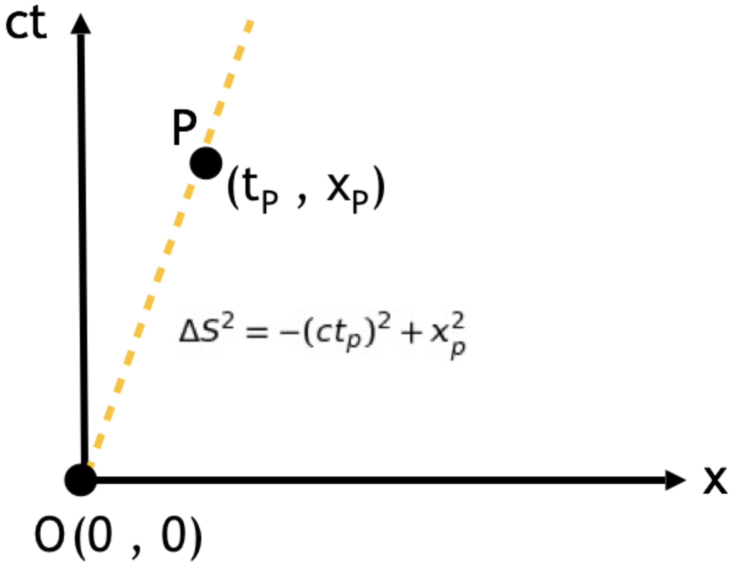
This **physical quantity** (物理量) is **invariant** (不變的) no matter you are talking about it in a **rest frame** or in a **moving reference frame**.

6.2 – Proper Time Intervals and Proper Lengths

In this section, we will show that the proper time intervals and proper lengths are hyperbolas (雙曲線) on the spacetime diagram.

Let's consider 2 events $O = O(0,0)$ and $P = P(t_P, x_P)$ in the rest inertial reference frame (S-frame). The **spacetime interval** between these 2 events in the S-frame will be:

$$\Delta S^2 = -[c(t_P - t)]^2 + (x_P - 0)^2 = -(ct_P)^2 + x_P^2$$



We define a new **parameter** (量) T such that:

$$\Delta S^2 = -(ct_P)^2 + x_P^2 = -(cT)^2$$

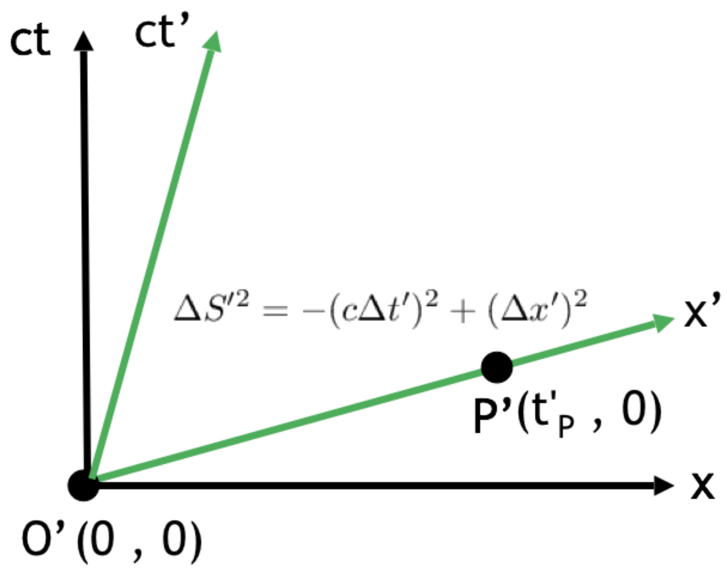
Now, we introduce a new moving observer S' such that he moves at a speed v_P given by

$$v_P^2 = \left(\frac{x_P}{t_P}\right)^2$$

Up till the present, we still have not successfully found anything which can move faster than the speed of light (or if you like, can travel back in time), so it is fair to suggest that

$$v_P^2 = \left(\frac{x_P}{t_P}\right)^2 < c^2$$

Then, in the S'-frame, we will have the spacetime coordinates of O (O') and P (P') as $O'(0,0)$ and $P'(t'_P, 0)$. Note that in this formulation, the S'-frame is actually **moving together** with an imaginary particle moving along **OP**.



The **spacetime interval** between the 2 events O and P' in the S'-frame will be:

$$\Delta S^2 = -[c(t'_P - 0)]^2 + (0-0)^2 = -(ct'_P)^2$$

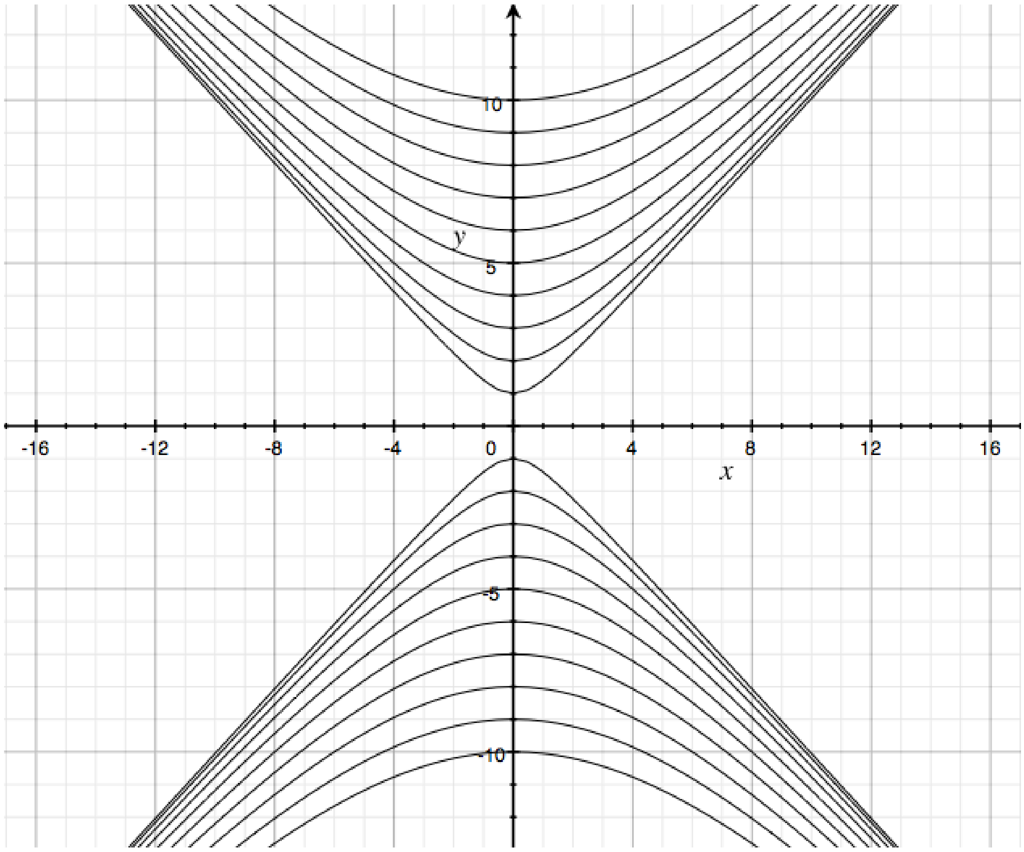
Then, making use of the **invariance** of spacetime interval, we have

$$\Delta S^2 = \Delta S'^2$$
$$-(cT)^2 = -(ct'_P)^2$$
$$T = t'_P$$

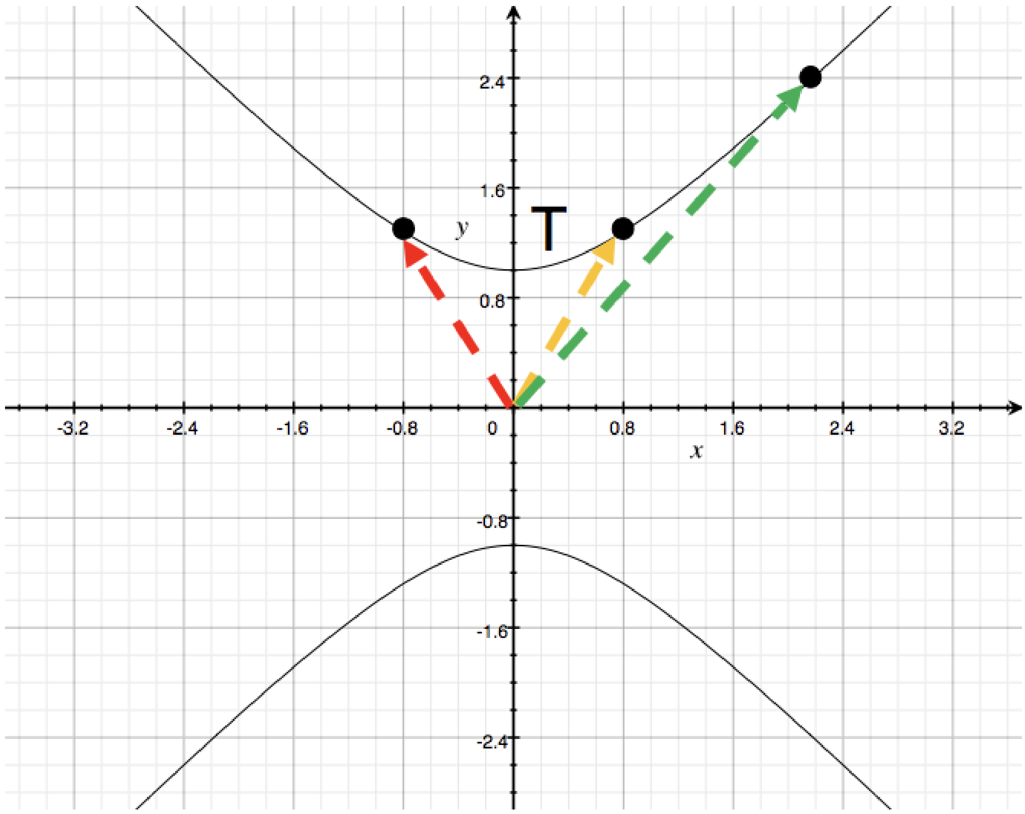
This shows that the **proper time interval** (本徵時間間隔) in the S'-frame **agrees with** that in the S-frame (i.e. The 2 proper "times" as measured the clocks in the S-frame and S'-frame are the same) as long as

$$-(ct')^2 = -(cT)^2 = -(ct)^2 + (x)^2$$

which, if we plot it on the spacetime diagram, are **hyperbolas** (雙曲線) along the ct axis.



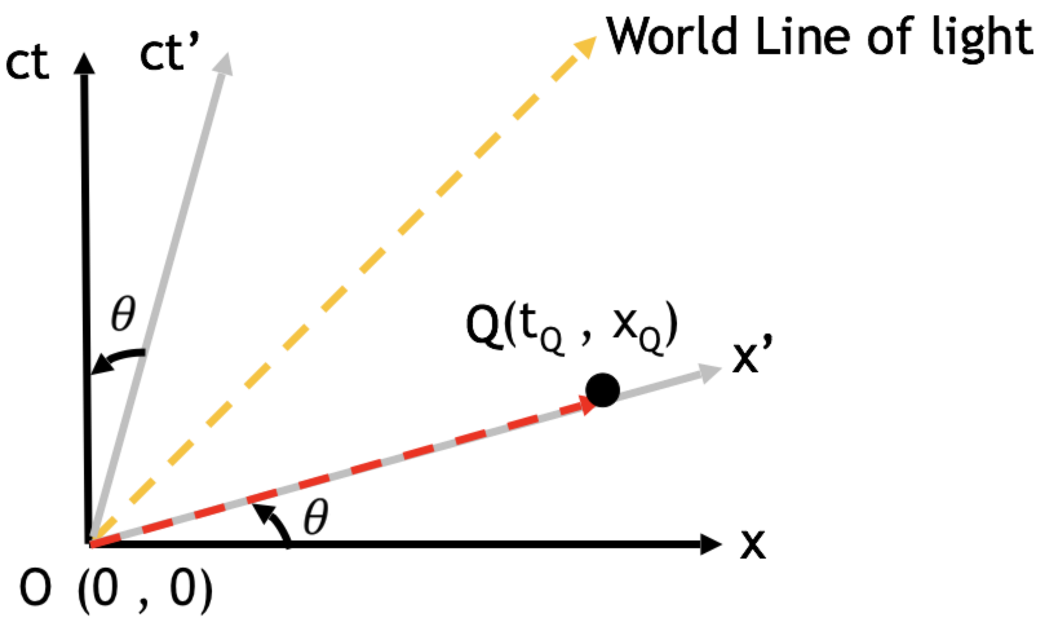
Along the hyperbolas, all observers (in ANY FRAME) will agree with the same proper time (i.e. $t = 1,2,3 \ldots$)



For example, the proper time measured by the RED, YELLOW and GREEN observers are all T (y-intercept of the hyperbola) as they move from O to the hyperbola.

This is quite the story for the "proper-time hyperbolas". Now we will move on to talk about **proper lengths**.

Similarly, we consider 2 events: $\mathrm{O} = (0,0)$ and $\mathrm{Q} = (\mathrm{t_Q}, \mathrm{x_Q})$ in the S-frame.



The **spacetime interval** between events O and Q in the S-frame will be

$$\Delta S^2 = -(c\Delta t)^2 + (\Delta x)^2 = -\left(ct_Q\right)^2 + x_Q^2$$

We define a new **parameter** (量) D such that:

$$\Delta S^2 = -(ct_P)^2 + x_P^2 = D^2$$

Now, we introduce a new moving observer S' such that he moves at a speed $\mathrm{v_Q}$ given by

$$\mathrm{v_Q^2} = \left(\frac{x_Q}{t_Q}\right)^2$$

Up till the present, we still have not successfully found anything which can move faster than the speed of light (or if you like, can travel back in time), so it is fair to suggest that

$$\mathrm{v_Q^2} = \left(\frac{x_Q}{t_Q}\right)^2 < c^2$$

With such formulation, we will have both events $\mathrm{O'} = (0,0)$ and $\mathrm{Q'} = (0, x_Q')$ happen

simultaneously in the S'-frame. Note that this is because the S'-frame moves at a speed fast enough such that O and Q happen simultaneously in his frame.

The **spacetime interval** between the 2 events O and Q' in the S'-frame will be:

$$\Delta S^2 = -[c(0-0)]^2 + \left(x'_Q - 0\right)^2 = {x'_Q}^2$$

Then, making use of the **invariance** of spacetime interval, we have

$$\Delta S^2 = \Delta S'^2$$

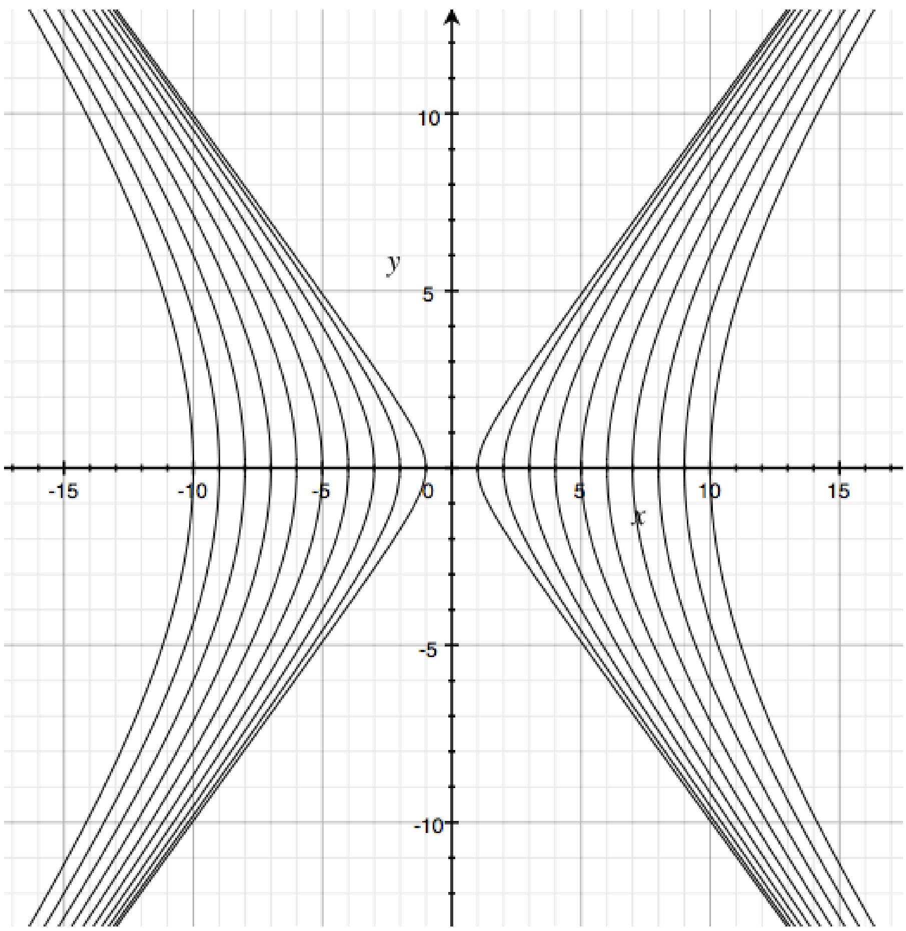
$$D^2 = {x'_Q}^2$$

$$D = x'_Q$$

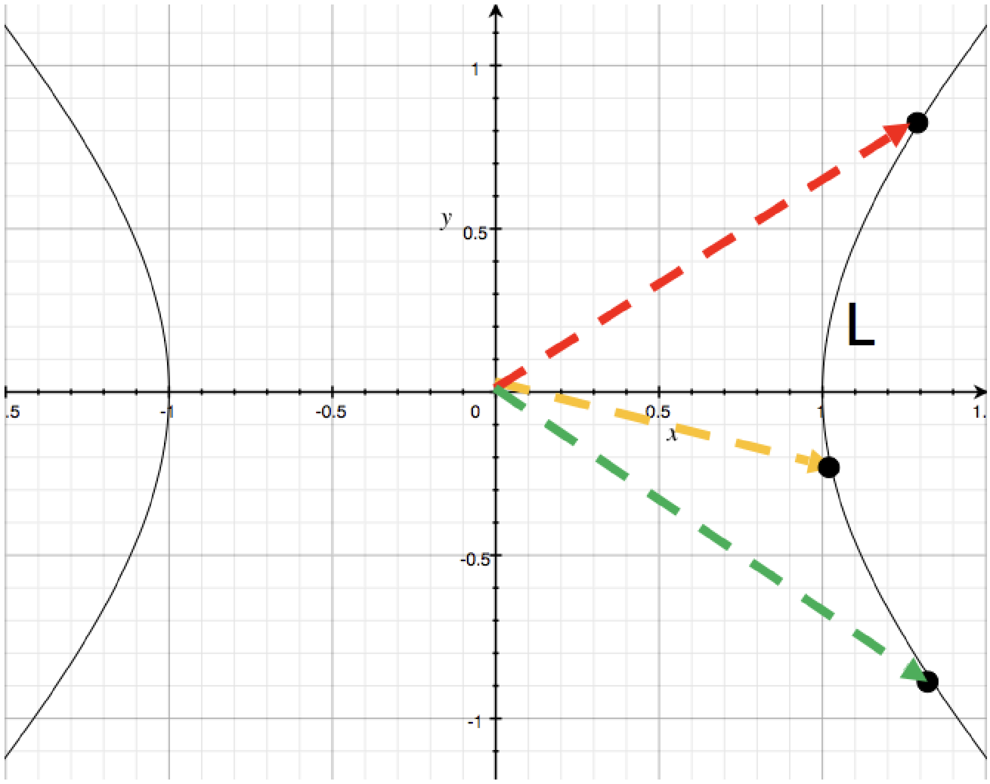
This shows that the **proper length interval** (本徵長度間隔) in the S'-frame **agrees with** that in the S-frame (i.e. The 2 proper "lengths" as measured the observers in the S-frame and S'-frame are the same) as long as

$$x'^2 = D^2 = -(ct)^2 + (x)^2$$

which, if we plot it on the spacetime diagram, are **hyperbolas** (雙曲線) along the x axis.

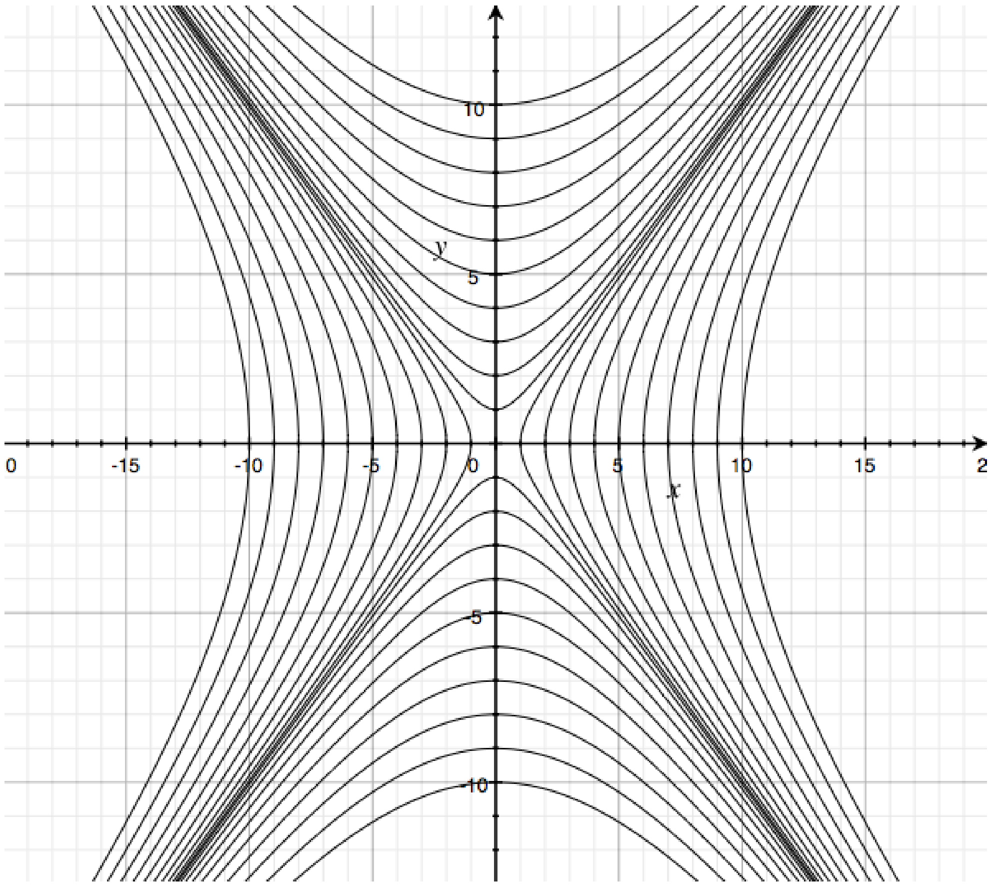


Along the hyperbolas, all observers (in ANY FRAME) will agree with the same proper lengths (i.e. $x = 1,2,3 \ldots$)



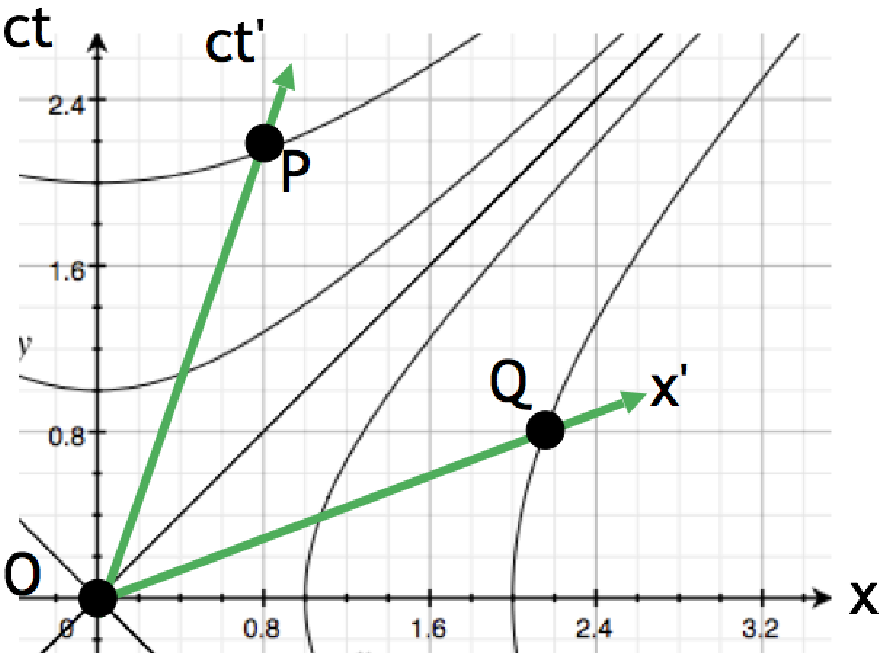
For example, the proper lengths measured by the RED, YELLOW and GREEN observers are all L (x-intercept of the hyperbola) as they move from O to the hyperbola.

The following diagram shows the **usual applicable form of spacetime diagram**.



Example 6.2

Consider the following spacetime diagram. An inertial rest frame (S-frame), a moving inertial frame (S'-frame) and 3 events – $O(0, 0)$, P and Q are shown.



(a) By considering **events** $\mathbf{O}$ **and** $\mathbf{P}$ **in the diagram,** show the <u>time dilation effect</u> *WITHOUT* doing any calculations.

(b) By considering **events** $\mathbf{O}$ **and** $\mathbf{Q}$ **in the diagram,** show the <u>length contraction effect</u> *WITHOUT* doing any calculations.

[Solutions]

(a) By considering **events** $\mathbf{O}$ **and** $\mathbf{P}$ **in the diagram,** show the <u>time dilation effect</u> *WITHOUT* doing any calculations.

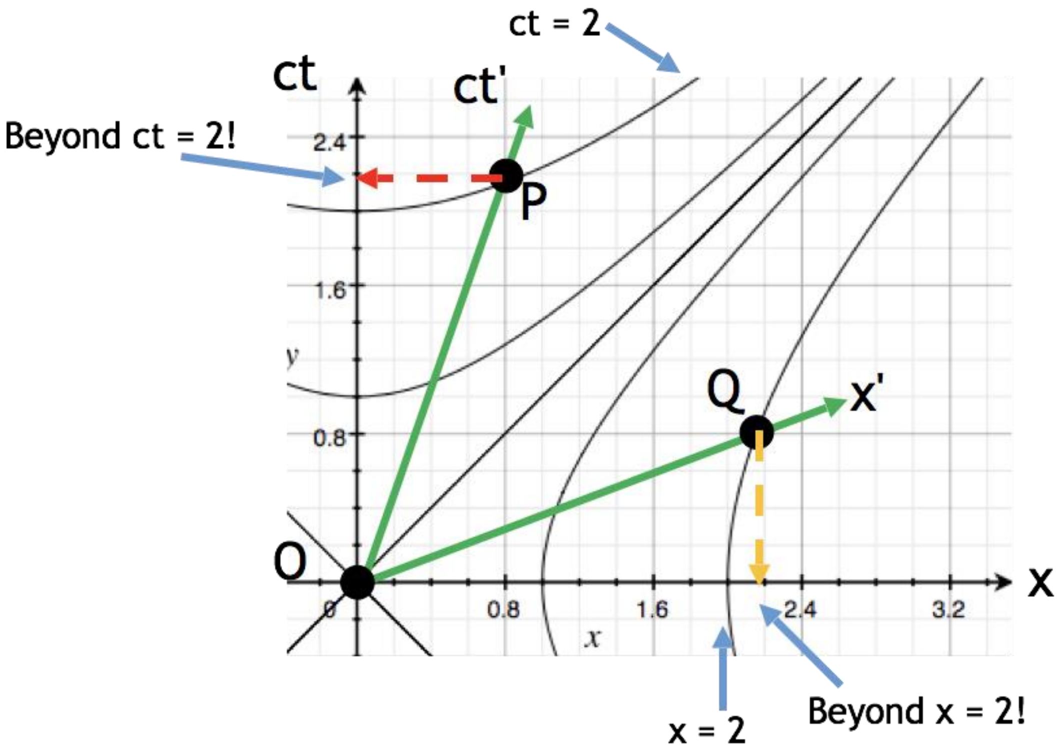
[Sol]

As seen in the figure below, the time coordinate of P in the S-frame is beyond $ct = 2$, while that in the S'-frame is still at $ct = 2$. This shows that "A moving clock (the proper time measured in the S'-frame) moves slower (than that in the S-frame)", which is the time dilation effect.

(b) By considering **events $\mathbf{O}$ and $\mathbf{Q}$ in the diagram,** show the <u>length contraction effect</u> ***WITHOUT*** doing any calculations.

[Sol]

As seen in the figure below, the space coordinate of P in the S-frame is beyond $\mathrm{x} = 2$, while that in the S'-frame is still at $\mathrm{x} = 2$. This shows that "A moving rod (the proper length measured in the S'-frame) contracts (compare to that in the S-frame)", which is the length contraction effect.



Challenge 6.2

Consider 2 inertial reference frames, S-frame and S'-frame in standard orientation (i.e. Both origins O and O' coincide at (0, 0)). The S'-frame moves with a velocity $0.6c$ along the x-axis relative to the S-frame. An event P occurs as $ct = 10$ and $x = 8$ in the S-frame.

(a) Sketch the following items on a standard hyperbola graph paper.

(i) The ct' and x' axes of the S'-frame.

(ii) The event P.

(b) Using your graph in (a), determine the time (ct') and space (x') coordinates of event P as seen from the S'-frame.

(c) Check that your results in (b) agrees with that you obtain by using the Lorentz Transformation equations.

Summary

Key Points

6.1 Spacetime interval

- "Spacetime interval" is analogous to "length" in usual Cartesian planes, but it refers to the "spacetime-difference between 2 events", and is defined by:
$$\Delta S^2 = -(c\Delta t)^2 + (\Delta x)^2$$
which is **invariant** in all inertial reference frames.

6.2 Proper Time Intervals and Proper Lengths

- Proper time intervals in different inertial reference frames agree as long as
$$-(ct')^2 = -(\mathrm{cT})^2 = -(\mathrm{ct})^2 + (x)^2$$
which are **hyperbolas** along the ct axis in the spacetime diagram.

- Proper lengths in different inertial reference frames agree as long as
$$x'^2 = \mathrm{D}^2 = -(\mathrm{ct})^2 + (x)^2$$
which are **hyperbolas** along the x axis in the spacetime diagram.

Key Terms

Anti-clockwise 逆時針地	P.1	Context 背景	P.3
Coordinate geometry 座標幾何	P.1	Deceive 欺騙	P.1
Event 事件	P.3	Fake 虛假的	P.4
Hyperbola 雙曲線	P.9	Invariant 不變的	P.2
Invariance 不變性	P.3	Parameter 量	P.8
Physical quantity 物理量	P.3	Proper time interval 本徵時間間隔	P.9
Proper length interval 本徵長度間隔	P.12	Pythagoras theorem 畢氏定理	P.3
Rotate 旋轉	P.1	Spacetime interval 時空區間	P.3
Transformation 轉換	P.2		

Check Your Concepts

1. What does it mean by “spacetime interval”? What properties do it have? [Section 6.1]
2. What is the mathematical expression for spacetime interval? [Section 6.1]
3. What do the hyperbolas along the ct-axis and the x-axis represent in a spacetime diagram? [Section 6.2]

Historical Profile

Karl Schwarzschild was a German physicist and astronomer. He was also the father of astrophysicist Martin Schwarzschild. He provided the first exact solution to the Einstein field equations of general relativity, for the limited case of a single spherical non-rotating mass, which he accomplished in 1915, the same year that Einstein first introduced general relativity. The Schwarzschild solution, which makes use of Schwarzschild coordinates and the Schwarzschild metric, leads to a derivation of the Schwarzschild radius, which is the size of the event horizon of a non-rotating black hole. Schwarzschild accomplished this while serving in the German army during World War I.

Chapter Exercise

Multiple Choice Questions

1. Two events $P(\frac{3}{c},5)$ and $Q(\frac{5}{c},7)$ are observed in an inertial rest frame S. Another inertial frame S' is moving relative to S at a speed of $0.8c$. Find the spacetime interval $\Delta S'^2$ between events P and Q as seen from the **S'-frame**.

   A. 8
   B. -8
   C. 0
   D. Not enough information is given to deduce the answer.

2. Which of the following(s) about "spacetime interval" is correct?

   (1) It is **the same** in all inertial reference frames.
   (2) It describes the **space-time** difference between events in space-time.
   (3) It is **path-independent**. (i.e. It is the same no matter if we evaluate it along a straight path or a curved path)

   A. (2) only
   B. (3) only
   C. (1) and (2) only
   D. (2) and (3) only

3. How do we call the lines described by the equation

$$x'^2 = -(ct)^2 + x^2$$

   where ct and x are the usual time and space coordinate, and x' is the space coordinate in another inertial reference frame?

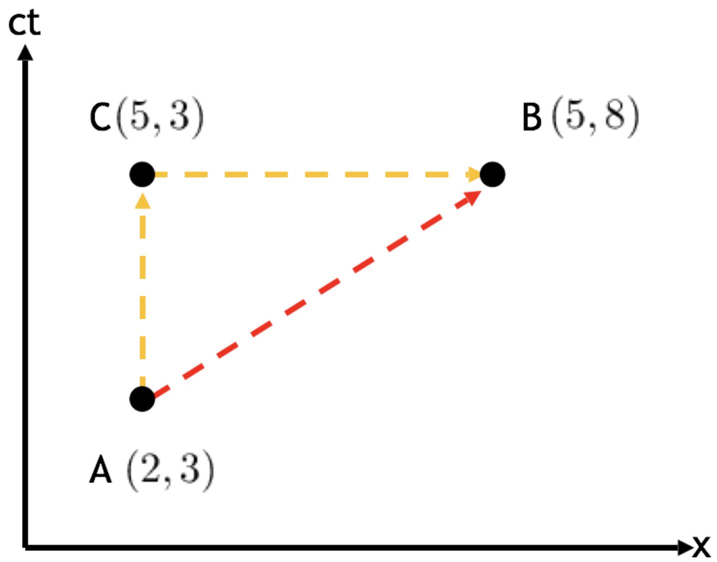
   A. Parabolas
   B. Circles
   C. Cycloids
   D. Hyperbolas

4. If 2 events have the same spacetime interval, which of the following must be correct?

   A. There is always a frame for which the 2 events happen simultaneously.
   B. There is always a frame for which the 2 events happen at the same place.
   C. The 2 events are actually the same. This is analogous to the **Uniqueness Theorem**.
   D. None of the above.

Short Questions

1. Consider the following spacetime diagram.



(a) Compute

(i) ΔS^2_{AC} (spacetime interval between A and C)

(ii) ΔS^2_{CB} (spacetime interval between C and B)

(iii) ΔS^2_{AB} (spacetime interval between A and B)

(b) What is $\Delta S^2_{AC} + \Delta S^2_{CB}$? Compare your answer with that in (a)(iii).

(c) Does your answer in (b) implies that **spacetime intervals** are **path-independent**? If yes, explain briefly. If no, can you try to suggest a counter-example?

Structured Questions

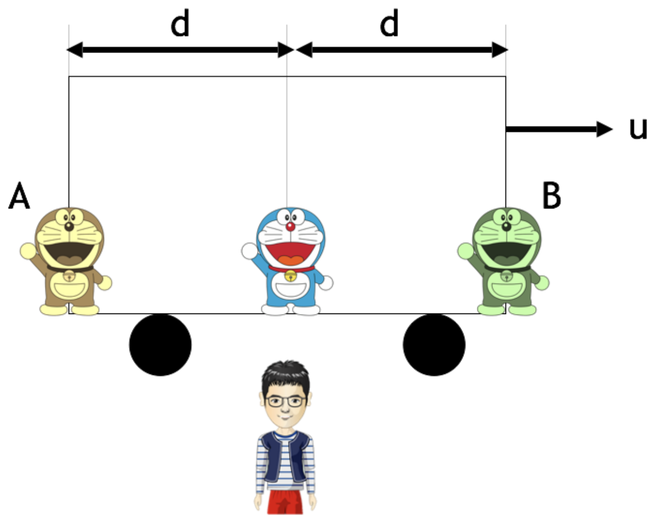
[Question 1] (Difficulty : ★★)

We have verified that spacetime interval can be described by the equation:

$$\Delta S^2 = -(c\Delta t)^2 + (\Delta x)^2$$

in **Example 6.1** and **Challenge 6.1** using 2 relatively simple approach. Now, we want to use a different approach to verify the result.

Consider the case below. A Doraemon is standing in the middle of a bus moving at a speed u to the right relative to the ground. Two of his friends, A and B, are at the left and right of the bus. Yiu Yung is standing outside the bus. At t = 0, Doraemon and Yiu Yung align in the same line in space.



(a) Sketch a spacetime diagram, showing the world lines of Doraemon, A, B and Yiu Yung. Take Yiu Yung's frame to be the S-frame (the rest inertial frame) and that of Doraemon to be the S'-frame. You may assume that the origin of the S-frame and S'-frame coincide.

(b) At $t = 0$, Doraemon sends 2 light signals simultaneously to A and B. Sketch the world lines of the 2 light signals on the same spacetime diagram. Mark the point at when the light signals reach A and B as events M and N respectively.

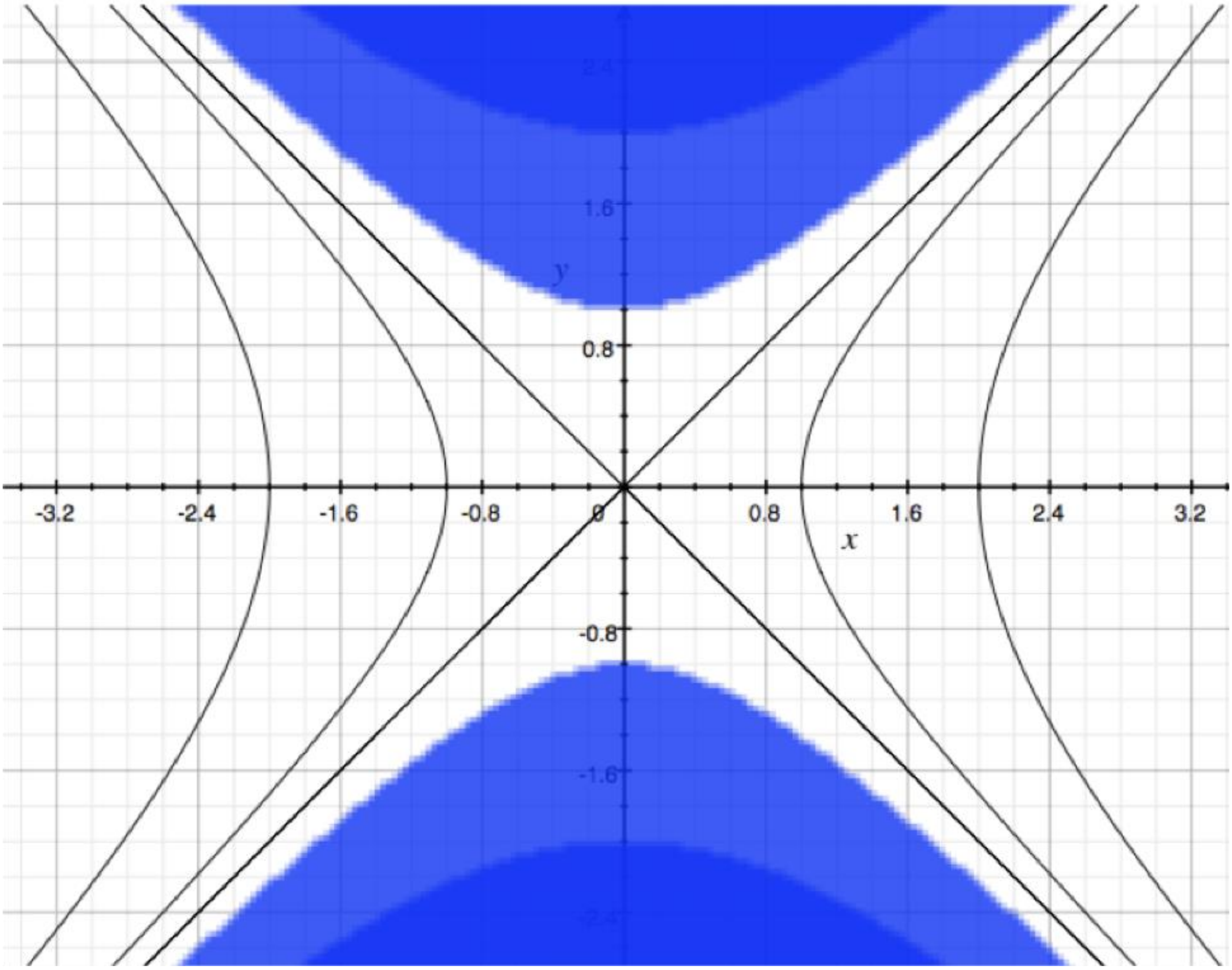
(c) Let the spacetime coordinates of event M and N be $M(t_1', x_1')$ and $N(t_2', x_2')$ as in the S'-frame. What is the relationship between t_1' and t_2'?

(d) Compute the spacetime interval between M and N in the S'-frame.

(e) Repeat (d) for the S frame. Hence show that spacetime interval is invariant in all inertial reference frame.

[Question 2] (Difficulty : ★★★)

In relativity, there is a theory claiming (and it is actually quite plausible) that there exists another universe which is almost disconnected from ours. We can illustrate it on a spacetime diagram.



The right hand side of the graph represents the usual universe we live in, while the left hand side represents the other parallel universe. The blue regions represents 2 important and yet up till today remaining-mysterious astronomical objects, the black-hole (upper) and white-hole (lower). They are **undefined regions** in which no one knows what is happening inside.

(a) Suppose you are standing at rest at $x = 1$. Sketch your world line on the diagram. Note that you **SHOULD NOT** extend your world line into the **undefined regions**.

(b) Suppose in the parallel universe, another you is also at rest at $x = -1$. Sketch his / her world line on the diagram. Note that you **SHOULD NOT** extend his / her world line into the **undefined region**.

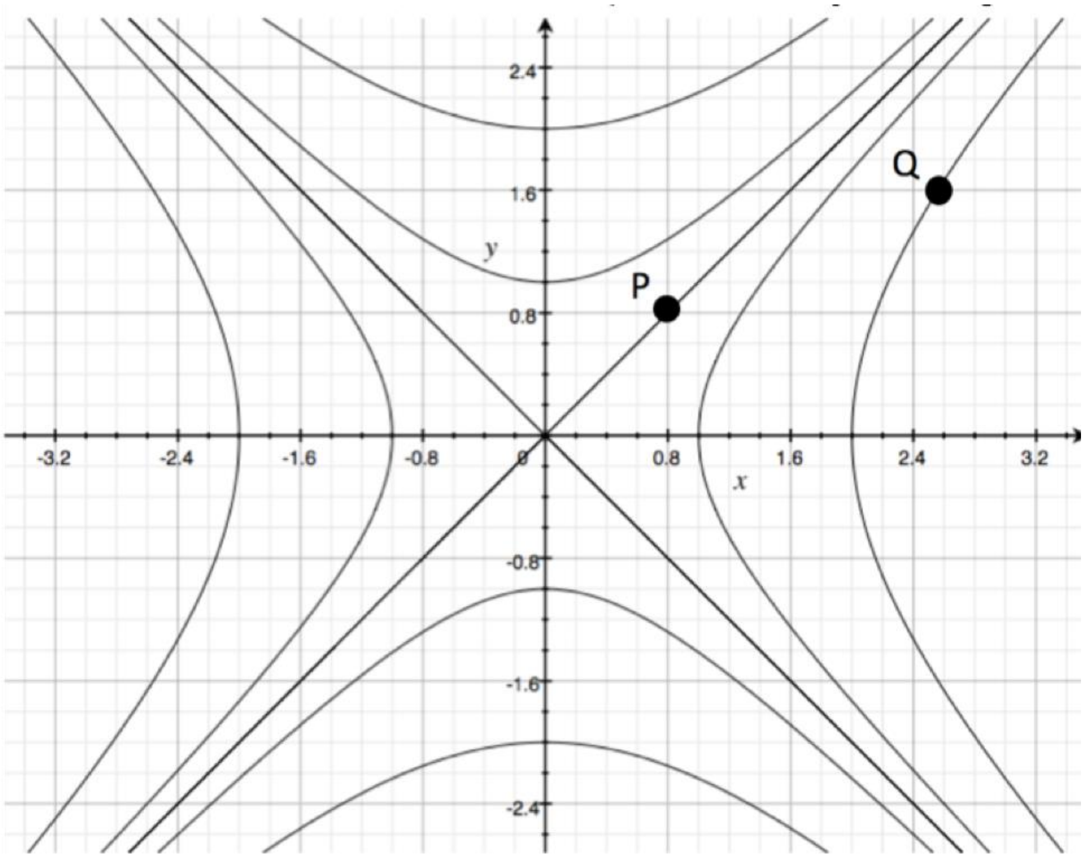
(c) Suppose at $t = 0$, you emit a light signal to your "clone" in the parallel universe. Sketch the world line of the signal. (**EXTEND** the world line to the undefined region using **dotted line**.) Ignore the undefined region, can the signal ever reach your clone?

(d) Suppose at $t = 0$, your clone emit a light signal to you from the parallel universe. Sketch the world line of the signal. (**EXTEND** the world line to the undefined region using **dotted line**.) Ignore the undefined region, can the signal ever reach your clone?

(e) In case both you and your clone fall into the undefined region, can you and your clone receive the signals? Show your answer graphically.

[Question 3] (Difficulty : ★★★★)

Refer to the spacetime diagram below. In a rest inertial frame S, there are 2 events P and Q.



(a) In the S-frame, which event, P or Q, happens first?

(b) Suppose time is **reversible** (可逆的). A man tries to travel back in time.

(i) If the man wants to see that the events P and Q happen in an reverse order compare to that in the S-frame. Suggest and sketch an appropriate pair of x' and ct' axes on the spacetime diagram for the man.

(ii) If we reflect P and Q along the x-axis, we will get 2 more events P' and Q'. In what order will P' and Q' happen in the time-traveller's frame you suggested in (b)(i)? How about the S-frame?

[Question 4] (Difficulty : ★★★★★)

Note : This question requires basic knowledge about **cylindrical and spherical coordinates**.

We have only been dealing with 1-D problems by now so far. In general, the spacetime interval in 3-D Cartesian coordinates can be described by:

$$\Delta S^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

We now want to express the 3-D spacetime interval using cylindrical coordinates and spherical coordinates.

(a) Using the fact that in cylindrical coordinates,

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$$

(i) Show that

$$\Delta x = (\Delta r)(\cos(\theta)) - r(\sin(\theta))\Delta\theta$$

<u>Need a helping hand?</u>

The Δ here means an infinitesimal change of something. You can regard it as differentials.

Suppose we have a function $f(x, y)$ consisting of 2 variables, we have

$$df = \frac{\partial f}{\partial x} \times dx + \frac{\partial f}{\partial y} \times dy$$

(ii) Show that

$$\Delta y = (\Delta r)(\sin(\theta)) + r(\cos(\theta))\Delta\theta$$

(iii) Using (a)(i) and (ii), show that the 3-D spacetime interval expressed in cylindrical coordinates is given by:

$$\Delta S^2 = -(c\Delta t)^2 + (\Delta r)^2 + (r\Delta\theta)^2 + (\Delta z)^2$$

(b) Using the fact that in spherical coordinates,

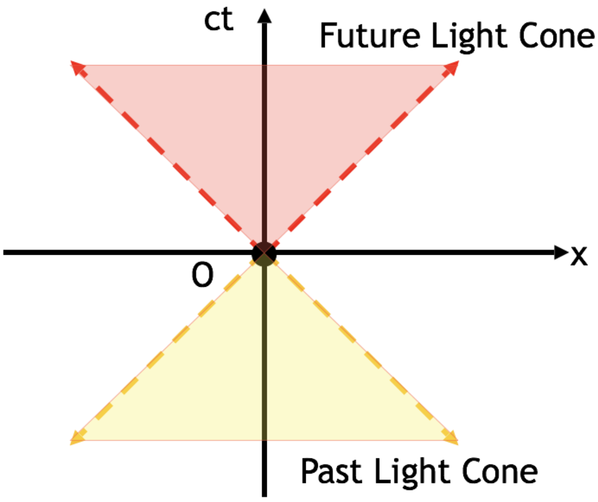
$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases}$$

Show that the 3-D spacetime interval expressed in spherical coordinates is given by

$$\Delta S^2 = -(c\Delta t)^2 + (\Delta r)^2 + (r\Delta\theta)^2 + (r\sin(\theta)\,\Delta\phi)^2$$

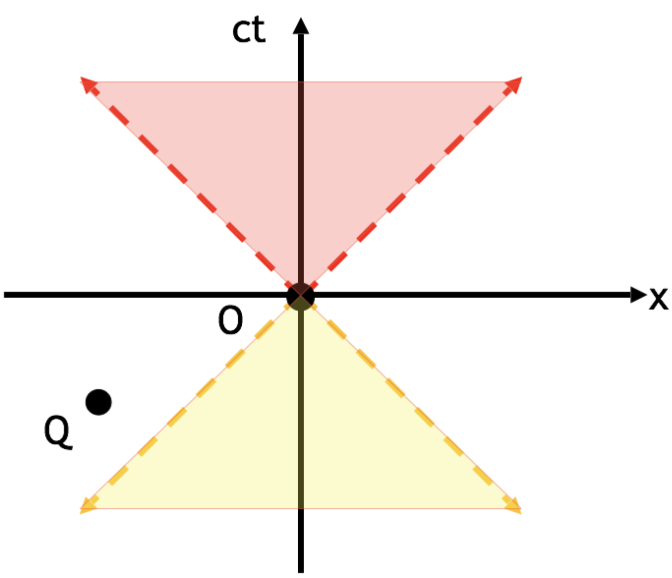
[Question 5] (Difficulty : ★★★ ★ ★★)

Let's consider a fixed point (the origin O) in spacetime. If we shoot 2 light rays from point O left and right, we will get 2 world lines of light signals as shown:



The red part refers to the region which is **future** relative to point O. While the yellow part is something that happens in the **past** relative to point O. We say the red part is the **future light cone**, and the yellow part is the **past light cone** relative to point O.

(a) Consider an event N which is neither in the **future light cone and past light cone** of point O as shown below. Show graphically that event N is never **influencing** (影響著) point O. Explain your answer briefly.



(b) What is the equation of the 2 world lines of light signals which passes through the point O in terms of ct and x? (Hint : Think about this in **Cartesian coordinates** – What is the equation of straight lines in the x-y plane making an angle $45°$ with the x-axis and passing through O?)

(c) Let's consider the future light cone (red part).

(i) For $\mathrm{x} < 0$,

(1) Write down an inequality describing the left-red region.

(2) Show that your answer in (1) can be written as:

$$-(\mathrm{ct})^2 + x^2 > 0$$

(Warning : This is no simple business. The reason is although $2 > -3$, but $(2)^2 = 4 \ngtr (-3)^2 = 9$. Think really carefully if you are doing mathematical-"legally")

(ii) For $\mathrm{x} > 0$,

(1) Write down an inequality describing the right-red region.

(2) Show that your answer in (1) can be written as:

$$-(\mathrm{ct})^2 + x^2 > 0$$

(d) Recall that spacetime interval is given by:

$$\Delta S^2 = -(c\Delta t)^2 + (\Delta x)^2$$

Using your answer in (c), show that

$$\Delta S^2 < 0$$

is a **critical condition** such that events can **influence** or **be influenced** by point O. Note that this is the same as saying that events must lie in the past or future light cones of O in order to influence or to be influence by point O.

(e) Can you guess the condition for events not to influence or to be influenced by O?

[Notes]

(1) For events which can influence each other, we say that they are **timelike-separated**, with the condition: $\Delta S^2 < 0$

(2) For events which cannot influence each other, we say that they are **spacelike-separated**, with the condition: $\Delta S^2 > 0$

(3) For events which can influence each other only by sending a light signal to each other (this is the only method), we say that they are **null-separated**, with the condition: $\Delta S^2 = 0$

~ The End~