Extending Superconformal Vector Fields on 6D Super-Minkowski Spacetime to 10D

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Abstract

The superconformal algebra's vector field representation of a $\mathcal{N} = (2, 0)$ superconformal theory in 6D found by Pär Arvidsson is reviewed and extended to 10D. It is shown that superconformal vector fields do not exist in 10D.

1 Introduction

The conformal field theories (CFT) are intensively studied nowadays because of their relation to string theories. For instance, in quantum gravity, there is a famous conjecture (AdS/CFT correspondence) that gravity theories in AdS spacetime can be understood by studying CFT on its boundary which looks like Minkowski spacetime. String theories also include supersymmetry, hence people like to study superconformal theories.

Superconformal vector fields are representations of some Lie superalgebra $A = A_0 \oplus A_1$ consisting of infinitesimal automorphisms on a super-Minkowski space that

1. extend the super-Poincaré algebra and
2. when their actions are restricted to ordinary Minkowski space, they reduce to the conformal algebra.

The Lie superbracket of the Lie superalgebra is written as $[\cdot, \cdot\} : A \times A \to A$ such that $[A, A_0\} := [A, A_0]$ and $[A_1, A_1\} := \{A_1, A_1\}$.

In this paper we discuss some results of a 6D superconformal theory before going into our discoveries about 10D superconformal algebra.

2 Conventions

2.1 Index notations

Latin letters in superscript or subscript on the σ^a matrices or space-time coordinates x^a denote the Lorentz space-time indices. In 6D, the Latin letters attached to fermionic coordinates θ^α_c and their partial derivatives ∂_α^c are the $\mathfrak{so}(5)$ R-symmetry indices. Greek letters denote the Lorentz spinor indices. Parentheses () and square brackets [ ] on superscript or subscript indices denote normalized symmetrization and anti-symmetrization over the indices concerned, respectively. For instance,

$$A_{[a}B_{b]} := \frac{1}{2}(A_aB_b - A_bB_a), \quad B_{(b}C_{c)} := \frac{1}{2}(B_bC_c + B_cC_b) \tag{2.1}$$

The bracket over the indices should not be confused with the commutator bracket, which is not normalized:

$$[A, B] := AB - BA \tag{2.2}$$

The Einstein summation convention was used. For instance, if $a \in \{0, ..., 9\}$, then

$$A^a B_{ab} = \sum_{a=0}^{9} A^a B_{ab} \tag{2.3}$$

2.2 6D Dirac matrices

Pauli matrices: $\tilde{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\tilde{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tilde{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\tilde{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
6D sigma matrices:

$$(\sigma^0)_{\alpha\beta} = i\tilde{\sigma}^2 \otimes \tilde{\sigma}^1, \tag{2.4}$$
$$(\sigma^1)_{\alpha\beta} = i\tilde{\sigma}^1 \otimes \tilde{\sigma}^2, \tag{2.5}$$
$$(\sigma^2)_{\alpha\beta} = \tilde{\sigma}^0 \otimes \tilde{\sigma}^2, \tag{2.6}$$
$$(\sigma^3)_{\alpha\beta} = -i\tilde{\sigma}^3 \otimes \tilde{\sigma}^2, \tag{2.7}$$
$$(\sigma^4)_{\alpha\beta} = -\tilde{\sigma}^2 \otimes \tilde{\sigma}^3, \tag{2.8}$$
$$(\sigma^5)_{\alpha\beta} = i\tilde{\sigma}^2 \otimes \tilde{\sigma}^0. \tag{2.9}$$

$(\sigma^a)^{\alpha\beta}$ are defined by the Clifford algebra relation:

$$(\sigma^a)_{\alpha\beta}(\sigma^b)^{\beta\gamma} + (\sigma^b)_{\alpha\beta}(\sigma^a)^{\beta\gamma} = 2\eta^{ab}\delta_\alpha{}^\gamma \tag{2.10}$$

where the Minkowski space-time metric is taken as $\eta_{ab} = DiagonalMatrix(-1, 1, 1, 1, 1, 1)$.
Relevant composite matrices:

$$(\sigma^{ab})_\alpha{}^\gamma := \frac{1}{2}\left[(\sigma^a)_{\alpha\beta}(\sigma^b)^{\beta\gamma} - (\sigma^b)_{\alpha\beta}(\sigma^a)^{\beta\gamma}\right] = (\sigma^{[a}\sigma^{b]})_\alpha{}^\gamma \tag{2.11}$$

2.3 10D Dirac matrices

We follow the convention used by Gates, Hu, Jiang, and Mak (2019):

$$(\sigma^0)_{\alpha\beta} = \tilde{\sigma}^0 \otimes \tilde{\sigma}^0 \otimes \tilde{\sigma}^0 \otimes \tilde{\sigma}^0, \tag{2.12}$$
$$(\sigma^1)_{\alpha\beta} = \tilde{\sigma}^2 \otimes \tilde{\sigma}^2 \otimes \tilde{\sigma}^2 \otimes \tilde{\sigma}^2, \tag{2.13}$$
$$(\sigma^2)_{\alpha\beta} = \tilde{\sigma}^2 \otimes \tilde{\sigma}^2 \otimes \tilde{\sigma}^0 \otimes \tilde{\sigma}^1, \tag{2.14}$$

$$(\sigma^3)_{\alpha\beta} = \tilde{\sigma}^2 \otimes \tilde{\sigma}^2 \otimes \tilde{\sigma}^0 \otimes \tilde{\sigma}^1, \tag{2.15}$$
$$(\sigma^4)_{\alpha\beta} = \tilde{\sigma}^2 \otimes \tilde{\sigma}^1 \otimes \tilde{\sigma}^2 \otimes \tilde{\sigma}^0, \tag{2.16}$$
$$(\sigma^5)_{\alpha\beta} = \tilde{\sigma}^2 \otimes \tilde{\sigma}^3 \otimes \tilde{\sigma}^2 \otimes \tilde{\sigma}^0, \tag{2.17}$$
$$(\sigma^6)_{\alpha\beta} = \tilde{\sigma}^2 \otimes \tilde{\sigma}^0 \otimes \tilde{\sigma}^1 \otimes \tilde{\sigma}^2, \tag{2.18}$$
$$(\sigma^7)_{\alpha\beta} = \tilde{\sigma}^2 \otimes \tilde{\sigma}^0 \otimes \tilde{\sigma}^3 \otimes \tilde{\sigma}^2, \tag{2.19}$$
$$(\sigma^8)_{\alpha\beta} = \tilde{\sigma}^1 \otimes \tilde{\sigma}^0 \otimes \tilde{\sigma}^0 \otimes \tilde{\sigma}^0, \tag{2.20}$$
$$(\sigma^9)_{\alpha\beta} = \tilde{\sigma}^3 \otimes \tilde{\sigma}^0 \otimes \tilde{\sigma}^0 \otimes \tilde{\sigma}^0 \tag{2.21}$$

Composite matrices:

$$(\sigma^{ab})_\alpha{}^\gamma := (\sigma^{[a}\sigma^{b]})_\alpha{}^\gamma, \tag{2.22}$$
$$(\sigma^{abc})_{\alpha\gamma} := (\sigma^{bc})_\alpha{}^\beta(\sigma^a)_{\beta\gamma} + 2\eta^{a[b}(\sigma^{c]})_{\alpha\gamma}, \tag{2.23}$$
$$(\sigma^{abcd})_\alpha{}^\gamma := (\sigma^a)_{\alpha\beta}(\sigma^{bcd})^{\beta\gamma} - 3\eta^{a[b}(\sigma^{cd]})_\alpha{}^\gamma, \tag{2.24}$$
$$(\sigma^{abcde})_{\alpha\gamma} := (\sigma^{bcde})_\alpha{}^\beta(\sigma^a)_{\beta\gamma} + 4\eta^{a[b}(\sigma^{cde]})_{\alpha\gamma} \tag{2.25}$$
$$\tag{2.26}$$

By the above definitions, we find that the matrices with two lower spinor indices can be classified into
Symmetric matrices: $(\sigma^a)_{\alpha\beta} = (\sigma^a)_{\beta\alpha}, (\sigma^{abcde})_{\alpha\beta} = (\sigma^{abcde})_{\beta\alpha}$
Anti-symmetric matrices: $(\sigma^{abc})_{\alpha\beta} = -(\sigma^{abc})_{\beta\alpha}$
The matrices with one lower and one upper spinor indices are spanned by the following basis: $\delta_\alpha{}^\beta, (\sigma^{[2]})_\alpha{}^\beta, (\sigma^{[4]})_\alpha{}^\beta$

3 6D $\mathcal{N} = (2, 0)$ superconformal algebra generators

We assume a Minkowski superspace which extends the 6D Minkowski space-time. It has ordinary 5+1 space-time coordinates x^a that commutes with other coordinates ($x^a\theta^\alpha = \theta^\alpha x^a, x^a x^b = x^b x^a$), and fermionic coordinates that anti-commutes with fermionic coordinates ($\theta^\alpha\theta^\beta = -\theta^\beta\theta^\alpha$). Together, these coordinates form a real vector space. The range of Lorentz space-time indices is $a = (0, 1, 2, 3, 4, 5)$, and the range of Lorentz spinor indices is $\alpha = (1, 2, 3, 4)$.

Arvidsson (2006) gave the vector field representations of the conformal subalgebra's bosonic generators in a 6D $\mathcal{N} = (2, 0)$ superconformal theory:

$$P_{\alpha\beta} = \partial_{\alpha\beta} \tag{3.1}$$
$$D = x^{\alpha\beta}\partial_{\alpha\beta} + \frac{1}{2}\theta^\alpha_a\partial^a_\alpha \tag{3.2}$$
$$M_\alpha{}^\beta = 2x^{\beta\gamma}\partial_{\alpha\gamma} - \frac{1}{2}\delta_\alpha{}^\beta x^{\gamma\delta}\partial_{\gamma\delta} + \theta^\beta_c\partial^c_\alpha - \frac{1}{4}\delta_\alpha{}^\beta\theta^\gamma_c\partial^c_\gamma \tag{3.3}$$
$$K^{\alpha\beta} = -4x^{\alpha\gamma}x^{\beta\delta}\partial_{\gamma\delta} - \theta^g_a\Omega^{ab}\theta^{[\alpha}_b\theta^{\beta]}_c\Omega^{cd}\theta^\delta_d\partial_{\gamma\delta} + 2\theta^{[\alpha}_c(2x^{\beta]\gamma} - i\theta^{\beta]}_a\Omega^{ab}\theta^\gamma_b)\partial^c_\gamma \tag{3.4}$$

where Ω^{ab} is the antisymmetric invariant tensor for R-symmetry transformation, and the partial derivatives are defined by $\partial_{\alpha\beta}x^{\gamma\delta} := \delta_{[\alpha}{}^\gamma\delta^\delta_{\beta]}$ and $\partial^b_\alpha\theta^\beta_c := \delta^b{}_c\delta_\alpha{}^\beta$. The spacetime coordinates are antisymmetric with respect to the Lorentz spinor indices: $x^{\alpha\beta} = -x^{\beta\alpha}$. Using the Dirac matrices in 6D, we can decompose the expressions above into

$$x^{\alpha\beta} = -\frac{1}{2}(\sigma^a)^{\alpha\beta}x_a \tag{3.5}$$

$$P_{\alpha\beta} = \frac{1}{2}(\sigma^a)_{\alpha\beta}P_a \tag{3.6}$$

$$D = x^a\partial_a + \frac{1}{2}\theta^\alpha_a\partial^a_\alpha \tag{3.7}$$

$$M_\alpha{}^\beta = \frac{1}{2}(\sigma^{ab})_\alpha{}^\beta M_{ab} \tag{3.8}$$

$$K^{\alpha\beta} = -\frac{1}{2}(\sigma^a)^{\alpha\beta}K_a \tag{3.9}$$

where P_a, M_{ab}, D, K_a are the generators of spacetime translations, Lorentz transformations, dilation, and the special conformal transformations, respectively. Hence, we get

$$P_a = \partial_a \tag{3.10}$$

$$M_{ab} = -\frac{1}{2}x_a\partial_b + \frac{1}{2}x_b\partial_a - \frac{1}{4}\theta^\beta_c(\sigma_{ab})_\beta{}^\alpha\partial^c_\alpha \tag{3.11}$$

$$\begin{aligned} K_a = {} & x^2\partial_a - 2x_a x\cdot\partial + \frac{1}{4}(\sigma_a)_{\beta\alpha}\theta^\alpha\cdot\theta^\gamma\theta^\beta\cdot\theta^\delta(\sigma_b)_{\gamma\delta}\partial_b - x_a\theta\cdot\partial \\ & - x^b\theta^\alpha_c(\sigma_{ab})_\alpha{}^\gamma\partial^c_\gamma + i(\sigma_a)_{\beta\alpha}\theta^\alpha_c\theta^\beta\cdot\theta^\gamma\partial^c_\gamma \end{aligned} \tag{3.12}$$

The dot products are defined by $x\cdot\partial := x^c\partial_c$, $\theta\cdot\partial := \theta^\alpha_c\partial^c_\alpha$, and $\theta^\alpha\cdot\theta^\beta := \theta^\alpha_a\Omega^{ab}\theta^\beta_b$. The four generators satisfy the conformal algebra relations (only the non-commuting ones are shown below):

$$[P_a, M_{bc}] = -\eta_{a[b}P_{c]} \tag{3.13}$$

$$[K_a, D] = -K_a \tag{3.14}$$

$$[P_a, D] = P_a \tag{3.15}$$

$$[K_a, M_{bc}] = -\eta_{a[b}K_{c]} \tag{3.16}$$

$$[M_{ab}, M_{cd}] = \eta_{c[a}M_{b]d} - \eta_{d[a}M_{b]c} \tag{3.17}$$

$$[P_a, K_b] = -4M_{ab} - 2\eta_{ab}D \tag{3.18}$$

The fermionic generators of the superconformal algebra are the supercharge Q^c_a and the special supersymmetry generator S^α_c, expressed as

$$Q^a_\alpha = \partial^a_\alpha - \frac{1}{2}i\Omega^{ac}\theta^\gamma_c(\sigma^b)_{\alpha\gamma}\partial_b \tag{3.19}$$

$$S^\alpha_a = -\Omega_{ac}((\sigma^b)^{\alpha\gamma}x_b + i\theta^\alpha\cdot\theta^\gamma)\partial^c_\gamma + 2i\theta^\gamma_a\theta^\alpha_c\partial^c_\gamma + \frac{1}{2}ix_b\theta^\gamma_a(\sigma^{db})_\gamma{}^\alpha\partial_d + \frac{1}{2}i\theta^\alpha_a x\cdot\partial - \frac{1}{2}(\sigma^d)_{\gamma\delta}\theta^\gamma_a\theta^\delta\cdot\theta^\alpha\partial_d \tag{3.20}$$

These two generators have similar properties: Q^a_α and S^α_a are the "square-roots" of P_a and K_a respectively. To be precise, they satisfy the following defining properties:

$$\{S^\alpha_a, S^\beta_b\} = -2i\Omega_{ab}K^{\alpha\beta} \tag{3.21}$$

$$\{Q^a_\alpha, Q^b_\beta\} = -2i\Omega^{ab}P_{\alpha\beta} \tag{3.22}$$

where the R-symmetry invariant tensor satisfies $\Omega_{ab}\Omega^{bc} = \delta_a{}^c$.

4 10D superconformal algebra generators

Notice that in 10D we can define Majorana-Weyl spinors with 16 real components, and in the 6D theory the spinors also have 16 degrees of freedom because there are 4 Lorentz spinor indices and 4 R-symmetry indices. Hence, it is natural to ask the following question: can we extend this vector field representation of 6D superconformal algebra to 10D?
First, we could try the most straightforward step: simply replace the 4×4 σ matrices (defined in Section 2.2) appearing in the 6D bosonic generators by the 16×16 σ matrices (defined in Section 2.3). Next, remove the R-symmetry indices and make the following change: $\theta^\alpha \cdot \theta^\beta \mapsto \theta^\alpha \theta^\beta$. These new expressions are

$$P_a = \partial_a \tag{4.1}$$
$$M_{ab} = -\frac{1}{2}x_a\partial_b + \frac{1}{2}x_b\partial_a - \frac{1}{4}\theta^\beta (\sigma_{ab})_\beta{}^\alpha \partial_\alpha \tag{4.2}$$
$$D = x \cdot \partial + \frac{1}{2}\theta \cdot \partial \tag{4.3}$$
$$K_a = x^2\partial_a - 2x_a x \cdot \partial - x_a\theta \cdot \partial - x^b\theta^\alpha (\sigma_{ab})_\alpha{}^\gamma \partial_\gamma \tag{4.4}$$

It is verified that commutation relations shown in the previous section are still satisfied by these new vector fields. One should know that the expressions of these vector fields are subject to change if they are incompatible with the fermionic generators, i.e. if their commutators do not result in some fixed forms. This issue will be discussed in detail in the next section.
These are probably not all of the bosonic generators in a superconformal algebra in 10D (if such superalgebra exists). There could be more vector fields arising from the commutation relations when we consider the fermionic generators, just like the U^{ab} generator in the 6D theory that comes from

$$\{Q^a_\alpha, S^\beta_b\} = i\delta_\alpha{}^\beta (\delta^a{}_b D - 4\Omega_{bc}U^{ac}) + 2i\delta^a{}_b M_\alpha{}^\beta \tag{4.5}$$

However, before we look for more bosonic generators, we should look for the two essential fermionic generators in 10D: Q_α and S^α. First, consider the supercharge Q_α. It is require to satisfy

$$\{Q_\alpha, Q_\beta\} = -(\sigma^a)_{\alpha\beta}P_a \tag{4.6}$$
$$[Q_\alpha, D] = \frac{1}{2}Q_\alpha. \tag{4.7}$$

Thus, its vector field expression can be easily derived:

$$Q_a = \partial_\alpha - \frac{1}{2}(\sigma^a)_{\alpha\beta}\theta^\beta\partial_a. \tag{4.8}$$

Just as the supercharge Q_α is the "square-root" of the translation P_a, the special supersymmetry generator S^α is the "square-root" of the special conformal generator K_a, and it should satisfy (as suggested by the work of Arvidsson(Arvidsson, 2006))

$$\{S^\alpha, S^\beta\} = (\sigma^a)^{\alpha\beta}K_a \tag{4.9}$$
$$[P_a, S^\alpha] = (\sigma^a)^{\alpha\beta}Q_\beta \tag{4.10}$$
$$[S^\alpha, D] = -\frac{1}{2}S^\alpha. \tag{4.11}$$

Furthermore, it should satisfy some compatibility conditions with K_a:

$$[Q_\alpha, K_a] = k(\sigma_a)_{\alpha\beta}S^\beta \tag{4.12}$$

for some $k \in \mathbb{C}$. It appeared that the construction of S^α and K_a form a loop: for each definition of S^α there is a corresponding set of equations that characterize the compatibility condition between these two generators. In the next section, the general forms of K_a and S^α that satisfy some of these conditions are derived.

5 Constraints on the form of K_a and S^α

We start by listing some (but not all) of the commutation relations that we want K_a and S^α to satisfy:

$$[S^\alpha, D] = -\frac{1}{2}S^\alpha \tag{5.1}$$

$$[P_a, S^\alpha] = (\sigma^a)^{\alpha\beta}Q_\beta \tag{5.2}$$

$$[K_a, D] = -K_a \tag{5.3}$$

Next, consider two lemmas.

Lemma 1. *Suppose* $[\theta^{\alpha_1}\cdots\theta^{\alpha_n}\partial_a, D] = -\frac{1}{2}\theta^{\alpha_1}\cdots\theta^{\alpha_n}\partial_a$, *then* $n = 3$

Proof.

$$[\theta^{\alpha_1}\cdots\theta^{\alpha_n}\partial_a, D] = [\theta^{\alpha_1}\cdots\theta^{\alpha_n}\partial_a, x\cdot\partial + \frac{1}{2}\theta\cdot\partial] \tag{5.4}$$

$$= \theta^{\alpha_1}\cdots\theta^{\alpha_n}(\partial_a x^b)\partial_b - \frac{1}{2}\theta^\mu\partial_\mu(\theta^{\alpha_1}\cdots\theta^{\alpha_n})\partial_a \tag{5.5}$$

$$= \frac{2-n}{2}\theta^{\alpha_1}\cdots\theta^{\alpha_n}\partial_a \tag{5.6}$$

$$= -\frac{1}{2}\theta^{\alpha_1}\cdots\theta^{\alpha_n}\partial_a \tag{5.7}$$

$$\implies n = 3 \tag{5.8}$$

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Lemma 2. *Suppose* $[\theta^{\alpha_1}\cdots\theta^{\alpha_n}\partial_\alpha, D] = -\frac{1}{2}\theta^{\alpha_1}\cdots\theta^{\alpha_n}\partial_\alpha$, *then* $n = 2$

Proof.

$$[\theta^{\alpha_1}\cdots\theta^{\alpha_n}\partial_\alpha, D] = [\theta^{\alpha_1}\cdots\theta^{\alpha_n}\partial_\alpha, \frac{1}{2}\theta\cdot\partial] \tag{5.9}$$

$$= \frac{1}{2}(\theta^{\alpha_1}\cdots\theta^{\alpha_n}(\partial_\alpha\theta^\mu)\partial_\mu - \theta^\mu\partial_\mu(\theta^{\alpha_1}\cdots\theta^{\alpha_n})\partial_\alpha) \tag{5.10}$$

$$= \frac{1-n}{2}\theta^{\alpha_1}\cdots\theta^{\alpha_n}\partial_\alpha \tag{5.11}$$

$$= -\frac{1}{2}\theta^{\alpha_1}\cdots\theta^{\alpha_n}\partial_a \tag{5.12}$$

$$\implies n = 2 \tag{5.13}$$

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Proposition 1. *The generalized expression of S^α satisfying*

$$[S^\alpha, D] = -\frac{1}{2}S^\alpha, \tag{5.14}$$

$$[P_a, S^\alpha] = (\sigma^a)^{\alpha\beta}Q_\beta \tag{5.15}$$

is given by

$$\begin{aligned} S^\alpha =& x^a(\sigma_a)^{\alpha\beta}Q_\beta + k_1\theta^\alpha\theta\cdot\partial + k_2(\sigma^{abc})_{\beta\gamma}(\sigma_{abc})^{\alpha\delta}\theta^\beta\theta^\gamma\partial_\delta + \\ & k_3(\sigma^{abc})_{\beta\gamma}(\sigma_{bc})_\delta{}^\alpha\theta^\delta\theta^\beta\theta^\gamma\partial_a + k_4(\sigma_{bcd})_{\beta\gamma}(\sigma^{abcd})_\delta{}^\alpha\theta^\delta\theta^\beta\theta^\gamma\partial_a \end{aligned} \tag{5.16}$$

Proof. Simple calculation shows that

$$[P_a, x^b(\sigma_b)^{\alpha\beta}Q_\beta] = (\sigma_a)^{\alpha\beta}Q_\beta. \tag{5.17}$$

Then we have

$$[P_a, S^\alpha - x^b(\sigma_b)^{\alpha\beta}Q_\beta] = 0, \tag{5.18}$$

which means that except the first term, S^α is independent of x^b. These terms must obey the constraints given lemma 1 and lemma 2, and the coefficients can contain σ matrices.

Firstly consider ∂_α terms. Notice that we only have one free index α. By Lemma 2, there are two Grassmann coordinates. If the index is on a Grassmann coordinate, then the expression should be $\theta^\alpha X_\beta{}^\gamma\theta^\beta\partial_\gamma$. The only basis with no spacetime index in 10D is identity matrix, so $X_\beta{}^\gamma = \delta_\beta{}^\gamma$, and the term is $\theta^\alpha\theta\cdot\partial$. If all indices of Grassmann are dummy, then the terms should be $X_{\beta\gamma}{}^{\alpha\delta}\theta^\beta\theta^\gamma\partial_\delta$. Since X must be antisymmetric on two lower spinor indices and have no free spacetime index, we can easily decompose it into $X_{\beta\gamma}{}^{\alpha\delta} = (\sigma^{abc})_{\beta\gamma}(\sigma_{abc})^{\alpha\delta}$.

Next, consider ∂_a terms. If the free spinor index is on a θ, then it should be $(X^a)_{\beta\gamma}\theta^\alpha\theta^\beta\theta^\gamma\partial_a$. Since we don't have an object with only two antisymmetric spinor indices and one free spacetime index, the free spinor index can't be on θ. Thus the terms must be $(X^a)_{\beta\gamma\delta}{}^\alpha\theta^\delta\theta^\beta\theta^\gamma\partial_a$. The decomposition of X is obviously $(X^a)_{\beta\gamma\delta}{}^\alpha = c_1(\sigma^{abc})_{\beta\gamma}(\sigma_{bc})_\delta{}^\alpha + c_2(\sigma_{bcd})_{\beta\gamma}(\sigma^{abcd})_\delta{}^\alpha$. The proof is done. ☑

Now consider the constraints on the vector field expression of K_a.

Lemma 3. *Suppose that*

$$[x^{a_1}..x^{a_n}\theta^{\alpha_1}...\theta^{\alpha_m}\partial_a, D] = -x^{a_1}..x^{a_n}\theta^{\alpha_1}...\theta^{\alpha_m}\partial_a, \quad n, m \in \mathbb{N} \tag{5.19}$$

then $(n, m) \in \{(0, 4), (1, 2), (2, 0)\}$.

Proof.

$$\begin{aligned} [x^{a_1}..x^{a_n}\theta^{\alpha_1}...\theta^{\alpha_m}\partial_a, x\cdot\partial + \frac{1}{2}\theta\cdot\partial] =& x^{a_1}..x^{a_n}\theta^{\alpha_1}...\theta^{\alpha_m}(\partial_a x^b)\partial_b - x^b\partial_b(x^{a_1}..x^{a_n})\theta^{\alpha_1}...\theta^{\alpha_m}\partial_a \\ & - \frac{1}{2}x^{a_1}..x^{a_n}\theta^\beta\partial_\beta(\theta^{\alpha_1}...\theta^{\alpha_m})\partial_a \end{aligned} \tag{5.20}$$

$$= (1 - n - \frac{m}{2})(x^{a_1}..x^{a_n}\theta^{\alpha_1}...\theta^{\alpha_m}\partial_a) \tag{5.21}$$

$$1 - n - \frac{m}{2} = -1 \implies 2n + m = 4 \tag{5.22}$$

Since n, m are natural numbers, the desired result follows. ☑

Lemma 4. *Suppose that*

$$[x^{a_1}..x^{a_n}\theta^{\alpha_1}...\theta^{\alpha_m}\partial_\beta, D] = -x^{a_1}..x^{a_n}\theta^{\alpha_1}...\theta^{\alpha_m}\partial_\beta, \quad n, m \in \mathbb{N} \tag{5.23}$$

then $(n, m) \in \{(0, 3), (1, 1)\}$.

Proof.

$$[x^{a_1}..x^{a_n}\theta^{\alpha_1}...\theta^{\alpha_m}\partial_\beta, x\cdot\partial + \frac{1}{2}\theta\cdot\partial] = \frac{1}{2}x^{a_1}..x^{a_n}\theta^{\alpha_1}...\theta^{\alpha_m}(\partial_\beta\theta^\gamma)\partial_\gamma - x^b\partial_b(x^{a_1}..x^{a_n})\theta^{\alpha_1}...\theta^{\alpha_m}\partial_\beta$$

$$- \frac{1}{2}x^{a_1}..x^{a_n}\theta^\beta\partial_\beta(\theta^{\alpha_1}...\theta^{\alpha_m})\partial_\beta \tag{5.24}$$

$$= \left(\frac{1}{2} - n - \frac{m}{2}\right)(x^{a_1}..x^{a_n}\theta^{\alpha_1}...\theta^{\alpha_m}\partial_a) \tag{5.25}$$

$$\frac{1}{2} - n - \frac{m}{2} = -1 \implies 2n + m = 3 \tag{5.26}$$

Since n, m are natural numbers, the desired result follows. ☑∎

Proposition 2. *Suppose that* $[K_a, D] = -K_a$, *then*

$$K_a = x^2\partial_a - 2x_a x\cdot\partial + c_1 x_a\theta\cdot\partial + c_2 x^b\theta^\alpha(\sigma_{ab})_\alpha{}^\beta\partial_\beta + c_3\theta^\alpha\theta^\beta\theta^\gamma(\sigma^{[3]})_{\alpha\beta}(\sigma_{a[3]})_\gamma{}^\delta\partial_\delta + c_4\theta^\alpha\theta^\beta\theta^\gamma(\sigma_{a[2]})_{\alpha\beta}(\sigma^{[2]})_\gamma{}^\delta\partial_\delta$$
$$+ c_5 x_b\theta^\alpha\theta^\beta(\sigma_a{}^{bc})_{\alpha\beta}\partial_c + c_6\theta^\alpha\theta^\beta\theta^\gamma\theta^\delta(\sigma_{[3]})_{\alpha\beta}(\sigma^{[3]})_{\gamma\delta}\partial_a + c_7\theta^\alpha\theta^\beta\theta^\gamma\theta^\delta(\sigma_{a[2]})_{\alpha\beta}(\sigma^{[2]b})_{\gamma\delta}\partial_b \tag{5.27}$$

where $c_1, ..., c_7 \in \mathbb{C}$.

Proof. We begin by acknowledging the fact that the action of D acts linearly on the summands in K_a. The first two terms of K_a are fixed by the requirement that K_a becomes the ordinary special conformal generator in ordinary spacetime, and they correspond to the $(n, m) = (2, 0)$ terms in Lemma 3. These two terms will not be discussed below.

Consider the terms with spacetime derivatives ∂_c, and let n be the order of spacetime coordinates and m be the order of Grassmann coordinates. By Lemma 3, we have $(n, m) = (1, 2)$ or $(n, m) = (0, 4)$. If $(n, m) = (1, 2)$, then the term must contain exactly one σ matrix with two lower antisymmetric spinor indices, which can only be $(\sigma^{[3]})_{\alpha\beta}$. The free spacetime index a can be on the partial derivative, spacetime coordinate, or the σ matrix. In this case, it can only be in the σ matrix for a nontrivial expression. Thus we have the c_5 term. If $(n, m) = (0, 4)$, then we must have two σ matrices with antisymmetric lower spinor indices:$(\sigma^{[3]})_{\alpha\beta}$ and $(\sigma_{[3]})_{\gamma\delta}$. If the free spacetime index is on the partial derivative, then we get the c_6 term. Otherwise, we get the c_7 term.

Now consider the terms with fermionic derivatives ∂_β. By Lemma 4, we have $(n, m) = (0, 3)$ or $(n, m) = (1, 1)$. If $(n, m) = (1, 1)$, then the spacetime free index is either on the single spacetime coordinate or on a single $(\sigma_{ab})_\alpha{}^\beta$. They give the c_1 and c_2 terms. If $(n, m) = (0, 3)$, then the free spacetime index must lie in a σ matrix. This matrix has either two lower spinor indices or one lower and one upper spinor index. If it has two lower spinor indices, it must be $(\sigma_{a[2]})_{\alpha\beta}$, and the two dummy indices will appear again on $(\sigma^{[2]})_\gamma{}^\delta$. Thus we obtain the c_4 term. If the free spacetime index lies in a σ matrix with a lower and an upper spinor index, then it could be $(\sigma_{ab})_\gamma{}^\delta$ or $(\sigma_{a[3]})_\gamma{}^\delta$. However, if it were $(\sigma_{ab})_\gamma{}^\delta$, then the term must also contain a factor of $(\sigma^b)_{\alpha\beta}$, which would vanish due to its contraction with $\theta^\alpha\theta^\beta\theta^\gamma$ that is antisymmetric with respect to $\alpha\beta$ indices. Hence, our only remaining choice gives $(\sigma_{a[3]})_\gamma{}^\delta$ with $(\sigma^{[3]})_{\alpha\beta}$, resulting in the c_3 term. We have exhausted all the possible cases. ☑▮

Lemma 5.

$$\theta^\alpha\theta^\beta\theta^\gamma(\sigma_{a[3]})_\gamma{}^\delta(\sigma^{[3]})_{\alpha\beta} = 0 \tag{5.28}$$

$$\theta^\alpha\theta^\beta\theta^\gamma(\sigma_{a[2]})_{\alpha\beta}(\sigma^{[2]})_\gamma{}^\delta = 0 \tag{5.29}$$

$$\theta^\alpha\theta^\beta\theta^\gamma\theta^\delta(\sigma_{[3]})_{\alpha\beta}(\sigma^{[3]})_{\gamma\delta} = 0 \tag{5.30}$$

$$\theta^\alpha\theta^\beta\theta^\gamma\theta^\delta(\sigma_{a[2]})_{\alpha\beta}(\sigma^{[2]b})_{\gamma\delta} = 0 \tag{5.31}$$

Proof. The results follow from the equations (A.3),(A.8),(A.9), and (A.12) in the Appendix. The detailed derivation is left as an exercise for the readers. ☑▮

Corollary 5.1. *We can shorten the results of Proposition 1 and Proposition 2 to*

$$S^\alpha = x^a(\sigma_a)^{\alpha\beta}Q_\beta + k_1\theta^\alpha\theta\cdot\partial + k_2(\sigma^{abc})_{\beta\gamma}(\sigma_{abc})^{\alpha\delta}\theta^\beta\theta^\gamma\partial_\delta \tag{5.32}$$

$$K_a = x^2\partial_a - 2x_a x\cdot\partial + c_1 x_a\theta\cdot\partial + c_2 x^b\theta^\alpha(\sigma_{ab})_\alpha{}^\beta\partial_\beta + c_3 x_b\theta^\alpha\theta^\beta(\sigma_a{}^{bc})_{\alpha\beta}\partial_c. \tag{5.33}$$

Proof. The result is a trivial consequence of Lemma 5. ☑▮

After the expressions of K_a and S^α are simplified, we try to verify the remaining compatibility condition mentioned at the beginning of this section.

6 Non-existence of S^α

The readers should convince themselves of the following numerical fact:

$$\exists a, b \in \{0, ..., 9\} : (\sigma_c)_{\alpha[\beta}(\sigma^{abc})_{\gamma\delta]} \neq 0 \tag{6.1}$$

This implies $\theta^\beta\theta^\gamma\theta^\delta(\sigma^c)_{\alpha\beta}(\sigma_{abc})_{\gamma\delta} \neq 0$. Then, we are prepared to present the following result.

Theorem 1. *Suppose K_a and S^α are given by the expressions in Corollary 5.1. Then $\forall k \in \mathbb{C}$: $[Q_\alpha, K_a] \neq k(\sigma_a)_{\alpha\beta}S^\beta$.*

Proof. Suppose, to the contrary, that $\exists k \in \mathbb{C} : [Q_\alpha, K_a] = k(\sigma_a)_{\alpha\beta}S^\beta$. Next, calculate the terms in $[Q_\alpha, K_a]$.

$$[\partial_\alpha - \frac{1}{2}(\sigma^c)_{\alpha\beta}\theta^\beta\partial_c, x^2\partial_a] = -(\sigma^c)_{\alpha\beta}\theta^\beta x_c\partial_a \tag{6.2}$$

$$[\partial_\alpha - \frac{1}{2}(\sigma^c)_{\alpha\beta}\theta^\beta\partial_c, -2x_a x\cdot\partial] = (\sigma_a)_{\alpha\beta}\theta^\beta x\cdot\partial + (\sigma^c)_{\alpha\beta}\theta^\beta x_a\partial_c \tag{6.3}$$

$$[\partial_\alpha - \frac{1}{2}(\sigma^c)_{\alpha\beta}\theta^\beta\partial_c, x_a\theta\cdot\partial] = x_a\partial_\alpha - \frac{1}{2}(\sigma_a)_{\alpha\beta}\theta^\beta\theta\cdot\partial + \frac{1}{2}x_a\theta^\beta(\sigma^c)_{\alpha\beta}\partial_c \tag{6.4}$$

$$\begin{aligned}[\partial_\alpha - \frac{1}{2}(\sigma^c)_{\alpha\beta}\theta^\beta\partial_c, x^b\theta^\gamma(\sigma_{ab})_\gamma{}^\delta\partial_\delta] &= x^b(\sigma_{ab})_\alpha{}^\beta\partial_\beta - \frac{1}{2}x^b\theta^\beta(\sigma_{ab}{}^c)_{\alpha\beta}\partial_c - \frac{1}{2}x^b\theta^\beta(\sigma_b)_{\alpha\beta}\partial_a + \frac{1}{2}\theta^\beta(\sigma_a)_{\alpha\beta}x\cdot\partial \\ &- \frac{5}{64}\theta^\beta\theta^\gamma(\sigma_{a[2]})_{\beta\gamma}(\sigma^{[2]})_\alpha{}^\delta\partial_\delta + \frac{1}{64}\theta^\beta\theta^\gamma(\sigma^{[3]})_{\beta\gamma}(\sigma_{a[3]})_\alpha{}^\delta\partial_\delta\end{aligned} \tag{6.5}$$

$$[\partial_\alpha - \frac{1}{2}(\sigma^c)_{\alpha\beta}\theta^\beta\partial_c, x^b\theta^\gamma\theta^\delta(\sigma_{ab}{}^d)_{\gamma\delta}\partial_d] = 2x^b\theta^\beta(\sigma_{ab}{}^c)_{\alpha\beta}\partial_c - \frac{1}{2}\theta^\beta\theta^\gamma\theta^\delta(\sigma^b)_{\alpha\beta}(\sigma_{ab}{}^c)_{\gamma\delta}\partial_c \tag{6.6}$$

Summing them up, we obtain

$$\begin{aligned}[Q_\alpha, K_a] = &\left[c_1 x_a \delta_\alpha^{\ \beta} + c_2 x^b(\sigma_{ab})_\alpha^{\ \beta}\right]\partial_\beta - \frac{1}{2}c_3\theta^\beta\theta^\gamma\theta^\delta(\sigma^b)_{\alpha\beta}(\sigma_{ab}{}^c)_{\gamma\delta}\partial_c \\ &+ \theta^\beta\left[\left(1+\frac{1}{2}c_2\right)\left(-x^b(\sigma_b)_{\alpha\beta}\delta_a^{\ c} + x^c(\sigma_a)_{\alpha\beta}\right) + \left(1+\frac{1}{2}c_1\right)x_a(\sigma^c)_{\alpha\beta} + \left(2c_3 - \frac{1}{2}c_2\right)x^b(\sigma_{ab}{}^c)_{\alpha\beta}\right]\partial_c \\ &+ \theta^\beta\theta^\gamma\left[-\frac{1}{2}c_1(\sigma_a)_{\alpha\beta}\delta_\gamma^{\ \delta} + \frac{1}{64}c_2(\sigma^{[3]})_{\beta\gamma}(\sigma_{a[3]})_\alpha^{\ \delta} - \frac{5}{64}c_2(\sigma_{a[2]})_{\beta\gamma}(\sigma^{[2]})_\alpha^{\ \delta}\right]\partial_\delta. \end{aligned} \tag{6.7}$$

On the other hand, we have

$$k(\sigma_a)_{\alpha\beta}S^\beta = k(\sigma_a)_{\alpha\beta}\left[x^b(\sigma_b)^{\beta\gamma}Q_\gamma + k_1\theta^\beta\theta\cdot\partial + k_2(\sigma^{[3]})_{\gamma\epsilon}(\sigma_{[3]})^{\beta\delta}\theta^\gamma\theta^\epsilon\partial_\delta\right] \tag{6.8}$$

$$\begin{aligned} =&k\left[x^b(\sigma_{ab})_\alpha^{\ \beta} + x_a\delta_\alpha^{\ \beta}\right]\partial_\beta - \frac{1}{2}k\left[x^b(\sigma_{ab}\sigma^c)_{\alpha\beta} + x_a(\sigma^c)_{\alpha\beta}\right]\theta^\beta\partial_c \\ &+ k\theta^\beta\theta^\gamma\left[k_1(\sigma_a)_{\alpha\beta}\delta_\gamma^{\ \delta} + k_2(\sigma^{[3]})_{\beta\gamma}(\sigma_a\sigma_{[3]})_\alpha^{\ \delta}\right]\partial_d \end{aligned} \tag{6.9}$$

$$\begin{aligned} =&k\left[x^b(\sigma_{ab})_\alpha^{\ \beta} + x_a\delta_\alpha^{\ \beta}\right]\partial_\beta - \frac{1}{2}k\left[x^b(\sigma^c{}_{ab})_{\alpha\beta} - x^b(\sigma_b)_{\alpha\beta}\delta_a^{\ c} + x^c(\sigma_a)_{\alpha\beta} + x_a(\sigma^c)_{\alpha\beta}\right]\theta^\beta\partial_c \\ &+ k\theta^\beta\theta^\gamma\left[k_1(\sigma_a)_{\alpha\beta}\delta_\gamma^{\ \delta} + k_2(\sigma^{[3]})_{\beta\gamma}(\sigma_{a[3]})_\alpha^{\ \delta} + 3k_2(\sigma_{a[2]})_{\beta\gamma}(\sigma^{[2]})_\alpha^{\ \delta}\right]\partial_\delta \end{aligned} \tag{6.10}$$

Consider the first two terms in $[Q_\alpha, K_a]$ and $k(\sigma_a)_{\alpha\beta}S^\beta$. Notice that $\delta_\alpha^{\ \beta}$ and $(\sigma_{ab})_\alpha^{\ \delta}$ are linearly independent. Hence $k = c_1 = c_2$. Next, notice that there is no term with three θ's in $k(\sigma_a)_{\alpha\beta}S^\beta$, and recall that $(\sigma_{ab}{}^c)_{[\gamma\delta}(\sigma^b)_{\beta]\alpha} \neq 0$. Thus, we conclude that $c_3 = 0$. Now consider and compare the coefficients of ∂_c. Notice that $(\sigma_{ab}{}^c)_{\alpha\beta}$ and $(\sigma^c)_{\alpha\beta}$ are linearly independent. Thus, we obtain $2c_3 - \frac{1}{2}c_2 = -\frac{1}{2}k$ and $1 + \frac{1}{2}c_1 = -\frac{1}{2}k$. Since $c_3 = 0$ and $c_1 = c_2 = k$, we have $k = c_2 = c_1 = -1$. Finally, we look at the coefficients of ∂_δ.

$$-\frac{1}{2}(\sigma_a)_{\alpha[\beta}\delta_{\gamma]}^{\delta} + \frac{1}{64}(\sigma^{[3]})_{\beta\gamma}(\sigma_{a[3]})_\alpha^{\ \delta} - \frac{5}{64}(\sigma_{a[2]})_{\beta\gamma}(\sigma^{[2]})_\alpha^{\ \delta} \tag{6.11}$$

$$= k_1(\sigma_a)_{\alpha[\beta}\delta_{\gamma]}^{\delta} + k_2(\sigma^{[3]})_{\beta\gamma}(\sigma_{a[3]})_\alpha^{\ \delta} + 3k_2(\sigma_{a[2]})_{\beta\gamma}(\sigma^{[2]})_\alpha^{\ \delta} \tag{6.12}$$

From the Appendix (A.14), we have

$$(\sigma_a)_{\alpha[\beta}\delta_{\gamma]}^{\delta} = \frac{1}{32}(\sigma_{a[2]})_{\beta\gamma}(\sigma^{[2]})_\alpha^{\ \delta} + \frac{1}{96}(\sigma^{[3]})_{\beta\gamma}(\sigma_{a[3]})_\alpha^{\ \delta} \tag{6.13}$$

Thus, we obtain

$$-\left(k_1 + \frac{1}{2}\right)\left[\frac{1}{32}(\sigma_{a[2]})_{\beta\gamma}(\sigma^{[2]})_\alpha^{\ \delta} + \frac{1}{96}(\sigma^{[3]})_{\beta\gamma}(\sigma_{a[3]})_\alpha^{\ \delta}\right] \tag{6.14}$$

$$= \left(k_2 - \frac{1}{64}\right)(\sigma^{[3]})_{\beta\gamma}(\sigma_{a[3]})_\alpha^{\ \delta} + \left(3k_2 + \frac{5}{64}\right)(\sigma_{a[2]})_{\beta\gamma}(\sigma^{[2]})_\alpha^{\ \delta} \tag{6.15}$$

$$\Longrightarrow \left(-k_1\frac{1}{96} - k_2 + \frac{1}{96}\right)(\sigma^{[3]})_{\beta\gamma}(\sigma_{a[3]})_\alpha^{\ \delta} = \left(3k_2 + \frac{3}{32} + k_1\frac{1}{32}\right)(\sigma_{a[2]})_{\beta\gamma}(\sigma^{[2]})_\alpha^{\ \delta} \tag{6.16}$$

Recall that $(\sigma^{[3]})_{\beta\gamma}(\sigma_{a[3]})_\alpha^{\ \delta}$ and $(\sigma_{a[2]})_{\beta\gamma}(\sigma^{[2]})_\alpha^{\ \delta}$ are linearly independent. Thus, their coefficients must be equal to zero. The LHS gives $96k_2 + k_1 = 1$, but the RHS gives $96k_2 + k_1 = -3$. This is a contradiction. Therefore, $\forall k \in \mathbb{C} : [Q_\alpha, K_a] \neq k(\sigma_a)_{\alpha\beta}S^\beta$ ☑

We have shown that superconformal vector fields does not exist in 10D.

7 Discussion and Conclusion

Our result actually agrees with the mathematical analysis of Shnider (1988, 4) which states that superconformal algebra does not exist in even dimensions $d > 6$. Notice that the proof we presented in the previous section actually relies on some Fierz identities listed in the Appendix. This suggests that these identities are actually determined by the dimension of spacetime in an interesting way such that the superalgebra will not exist in odd dimensions.
Further calculations seem to imply that a more general expression of S^α under fewer restrictive assumptions still fails to satisfy $\{S^\alpha, S^\beta\} = k(\sigma^a)^{\alpha\beta} K_a$. For instance, we claim that any fermionic vector field S^α with units of $\sqrt{x}$ will not satisfy $\{S^\alpha, S^\beta\} = k(\sigma^a)^{\alpha\beta} K_a$. Since the essence of the problem is already presented in the previous sections, we will not include the details of other proofs for the non-existence of superconformal vector fields here.

A Appendix

Here are some 10D Fierz identities.

$$(\sigma^b)_{\alpha[\beta}(\sigma_{ab})_{\gamma]}{}^{\delta} = \frac{5}{32}(\sigma^{[2]})_{\alpha}{}^{\delta}(\sigma_{a[2]})_{\beta\gamma} - \frac{1}{32}(\sigma_{a[3]})_{\alpha}{}^{\delta}(\sigma^{[3]})_{\beta\gamma} \tag{A.1}$$

$$(\sigma^{[3]}\sigma^a)_{\gamma}{}^{(\alpha}(\sigma_{[3]})^{\beta)\delta} = -3(\sigma_b)^{\alpha\beta}(\sigma^{ab})_{\gamma}{}^{\delta} + \frac{1}{8}(\sigma^{a[4]})^{\alpha\beta}(\sigma_{[4]})_{\gamma}{}^{\delta} - 45(\sigma^a)^{\alpha\beta}\delta_{\gamma}{}^{\delta} \tag{A.2}$$

$$(\sigma^{a[3]})_{[\delta}{}^{\alpha}(\sigma_{[3]})_{\beta]\gamma} = -\frac{1}{2}(\sigma_{[3]})_{\delta\beta}(\sigma^{a[3]})_{\gamma}{}^{\alpha} \tag{A.3}$$

$$(\sigma^{a[3]})_{\delta}{}^{[\alpha}(\sigma_{[3]})^{\beta]\gamma} = -\frac{1}{2}(\sigma_{[3]})^{\alpha\beta}(\sigma^{a[3]})_{\delta}{}^{\gamma} \tag{A.4}$$

$$(\sigma_{cde}\sigma_{ab})^{[\beta\alpha]} = 2\eta_{c[b}(\sigma_{a]de})^{\alpha\beta} - 2\eta_{d[b}(\sigma_{a]ce})^{\alpha\beta} + 2\eta_{e[b}(\sigma_{a]cd})^{\alpha\beta} \tag{A.5}$$

$$(\sigma_{ab}\sigma_{cde})_{[\alpha\beta]} = 2\eta_{c[a}(\sigma_{b]de})_{\beta\alpha} - 2\eta_{d[a}(\sigma_{b]ce})_{\beta\alpha} + 2\eta_{e[a}(\sigma_{b]cd})_{\beta\alpha} \tag{A.6}$$

$$(\sigma_{[3]}\sigma_{abc})_{\gamma}{}^{(\alpha}(\sigma^{[3]})^{\beta)\delta} = 3(\sigma_d)^{\alpha\beta}(\sigma^{abcd})_{\delta}{}^{\delta} + \frac{3}{2}(\sigma^{abc[2]})^{\alpha\beta}(\sigma_{[2]})_{\gamma}{}^{\delta} - 9(\sigma^{[a})^{\alpha\beta}(\sigma^{bc]})_{\gamma}{}^{\delta} - \frac{3}{2}(\sigma^{[3][ab})^{\alpha\beta}(\sigma^{c]}{}_{[3]})_{\gamma}{}^{\delta} \tag{A.7}$$

$$(\sigma^{a[2]})_{\beta[\gamma}(\sigma_{[2]})_{\delta]}{}^{\alpha} = -\frac{1}{2}(\sigma_{[2]})_{\beta}{}^{\alpha}(\sigma^{a[2]})_{\gamma\delta} \tag{A.8}$$

$$(\sigma^{[3]})_{\alpha[\beta}(\sigma_{[3]})_{\gamma]\delta} = -\frac{1}{2}(\sigma^{[3]})_{\beta\gamma}(\sigma_{[3]})_{\alpha\delta} \tag{A.9}$$

$$[\sigma_{ab}, \sigma_{cd}] = -4\eta_{b[c}\sigma_{d]a} + 4\eta_{a[c}\sigma_{d]b} \tag{A.10}$$

$$(\sigma_c)_{\alpha[\beta}(\sigma^{abc})_{\gamma]\delta} = \frac{3}{8}(\sigma^{abc})_{\beta\gamma}(\sigma_c)_{\alpha\delta} - \frac{1}{4}(\sigma^{cd[a})_{\beta\gamma}(\sigma^{b]}{}_{cd})_{\alpha\delta} - \frac{1}{48}(\sigma_{[3]})_{\beta\gamma}(\sigma^{ab[3]})_{\alpha\delta} \tag{A.11}$$

$$(\sigma^{a[2]})_{\alpha[\beta}(\sigma^{b}{}_{[2]})_{\gamma]\delta} = \frac{7}{4}(\sigma^{abc})_{\beta\gamma}(\sigma_c)_{\alpha\delta} - \frac{1}{24}(\sigma_{[3]})_{\beta\gamma}(\sigma^{ab[3]})_{\alpha\delta} - \frac{1}{4}(\sigma^{a[2]})_{\beta\gamma}(\sigma^{b}{}_{[2]})_{\alpha\delta} - \frac{1}{4}(\sigma^{b[2]})_{\beta\gamma}(\sigma^{a}{}_{[2]})_{\alpha\delta} \tag{A.12}$$

$$(\sigma^{ab[3]})_{\alpha[\beta}(\sigma_{[3]})_{\gamma]\delta} = -\frac{21}{4}(\sigma^{abc})_{\beta\gamma}(\sigma_c)_{\alpha\delta} + \frac{3}{2}(\sigma^{[2][a})_{\beta\gamma}(\sigma^{b]}{}_{[2]})_{\alpha\delta} - \frac{3}{8}(\sigma_{[3]})_{\beta\gamma}(\sigma^{ab[3]})_{\alpha\delta} \tag{A.13}$$

$$(\sigma_a)_{\alpha\beta}\delta_{\gamma}{}^{\delta} = \frac{1}{16}(\sigma_a)_{\beta\gamma}\delta_{\alpha}{}^{\delta} + \frac{1}{16}(\sigma^b)_{\beta\gamma}(\sigma_{ab})_{\alpha}{}^{\delta} + \frac{1}{32}(\sigma_{a[2]})_{\beta\gamma}(\sigma^{[2]})_{\alpha}{}^{\delta} + \frac{1}{96}(\sigma^{[3]})_{\beta\gamma}(\sigma_{a[3]})_{\alpha}{}^{\delta} + \frac{1}{384}(\sigma_{a[4]})_{\beta\gamma}(\sigma^{[4]})_{\alpha}{}^{\delta} \tag{A.14}$$

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