SURE: Exploring the relation between Lie algebra, Supersymmetry, and Adindra

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Program:
Summer Student Theoretical physics Research Session (SSTPRS)

- Took place in June and last for a month.
- Mostly with US undergraduate students.

About the program

Learn some knowledge before doing research:
Group theory, Lie algebra, Tensor analysis, Riemann geometry, Supermultiplets

Research

Topic: Exploring the relation between Lie algebra, Supersymmetry, and Adindra

- Lie algebra
- Supermultiplets,
- Adinkra

Lie algebra: Special unitary group SU(2)

SU(n) Group: $n \times n$ unitary matrices with determinant 1.

su(2): Number of states: $N_{su(2)} = 2j + 1$
Pauli matrices ($j = \frac{1}{2}$):

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Take $(\frac{1}{2}\sigma_i)$ to be the generators:
Lie algebra:
$[(\frac{1}{2}\sigma_i), (\frac{1}{2}\sigma_j)] = i\epsilon_{ijk}(\frac{1}{2}\sigma_k)$

Weight space

The eigenvalues of $\frac{1}{2}\sigma_3$ form a weight space:

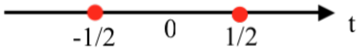


Figure 1: Weight space of su(2):

Lie algebra: Special unitary group SU(3)

SU(3): Number of states given by Weyl dimension formula:
$N_{su(3)} = \frac{1}{2}(p+1)(q+1)(p+q+2)$

Gell-Mann matrices (p=1,q=0):

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Let $\frac{1}{2}\lambda_i$ to be the generators:
Lie algbra: $[(\frac{1}{2}\lambda_i), (\frac{1}{2}\lambda_j)] = if_{ijk}(\frac{1}{2}\lambda_k)$

Lie algebra: Special unitary group SU(3)

Weight space

Commuting matrices:

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$(\tfrac{1}{2}\lambda_3)\,|u\rangle = \tfrac{1}{2}\,|u\rangle$$

$$(\tfrac{1}{2}\lambda_8)\,|u\rangle = \tfrac{1}{2\sqrt{3}}\,|u\rangle$$

A pair of eigenvalues: $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$

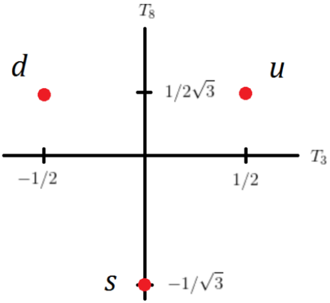


Figure 2: Weight space p=1, q=0

Lie algebra: Special unitary group SU(3)

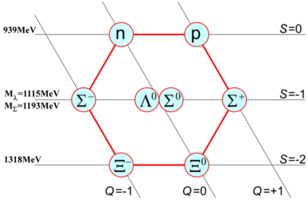


Figure 3: p=1, q=1

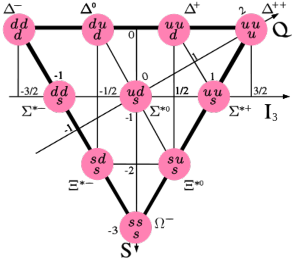


Figure 4: p=3, q=0

Minimal On-shell Supermultiplets of 4D N=1

D_a: Supercovariant derivative: Bosons$\longleftrightarrow$ *Fermions*

Chiral supermultiplet (A, B, ψ_c, F, G)

$$\begin{aligned} \mathrm{D}_a A =& \psi_a \qquad\qquad \mathrm{D}_a B = i(\gamma^5)_a{}^b \psi_b \\ \mathrm{D}_a \psi_b =& i(\gamma^\mu)_{ab}\partial_\mu A - (\gamma^5\gamma^\mu)_{ab}\partial_\mu B - iC_{ab}F + (\gamma^5)_{ab}G \\ \mathrm{D}_a F =& (\gamma^\mu)_a{}^b \partial_\mu \psi_b \qquad \mathrm{D}_a G = i(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \psi_b \end{aligned} \tag{1}$$

Vector supermultiplet (A_μ, λ_c, d)

$$\begin{aligned} \mathrm{D}_a A_\mu =& (\gamma_\mu)_a{}^b \lambda_b \qquad \mathrm{D}_a d = i(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \lambda_b \\ \mathrm{D}_a \lambda_b =& -i\tfrac{1}{4}([\gamma^\mu, \gamma^\nu])_{ab}(\partial_\mu A_\nu - \partial_\nu A_\mu) + (\gamma^5)_{ab} d \end{aligned} \tag{2}$$

Tensor supermultiplet $(\varphi, B_{\mu\nu}, \chi_c)$

$$\begin{aligned} \mathrm{D}_a \varphi =& \chi_a \qquad\qquad \mathrm{D}_a B_{\mu\nu} = -\tfrac{1}{4}([\gamma_\mu, \gamma_\nu])_a{}^b \chi_b \\ \mathrm{D}_a \chi_b =& i(\gamma^\mu)_{ab}\partial_\mu \varphi - \epsilon_\mu{}^{\rho\alpha\beta}(\gamma^5\gamma^\mu)_{ab}\partial_\rho B_{\alpha\beta} \end{aligned} \tag{3}$$

They all satisfy the *supersymmetry algebra*

$$\{\mathrm{D}_a, \mathrm{D}_b\} = i2(\gamma^\mu)_{ab}\partial_\mu \tag{4}$$

Figure 5: Supermultiplets: Chiral, Vector, Tensor

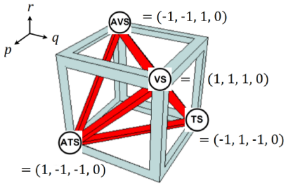
A tetrahedral appears

Supersymmetry algebra:$\{D_a, D_b\}$
Define holoraumy tensor:$[D_a, D_b]\mathcal{F}_c \equiv [H^{\mu(R)}]_{abc}{}^d(\partial_\mu \mathcal{F}_d)$,
where $\mathcal{F}$ is the fermionic field.

For 4D, N=1 minimal supermultiplets:

$$[\boldsymbol{H}^\mu(p_{(\mathcal{R})}, q_{(\mathcal{R})}, r_{(\mathcal{R})}, s_{(\mathcal{R})})]_{abc}{}^d = -i2[p_{(\mathcal{R})}C_{ab}(\gamma^\mu)_c{}^d + q_{(\mathcal{R})}(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d + r_{(\mathcal{R})}(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d + \frac{1}{2}s_{(\mathcal{R})}(\gamma^5\gamma^\nu)_{ab}(\gamma^5[\gamma_\nu, \gamma^\mu])_c{}^d]$$

$(\widehat{\mathcal{R}})$	$p_{(\mathcal{R})}$	$q_{(\mathcal{R})}$	$r_{(\mathcal{R})}$	$s_{(\mathcal{R})}$
(CS)	0	0	0	1
(VS)	1	1	1	0
(TS)	- 1	1	- 1	0



$(CS) = (0, 0, 0, 1)$

0-brane reduction: Only time dependent

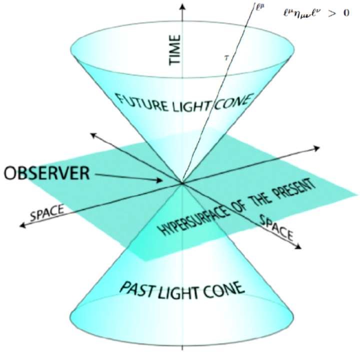


Figure 6: Lorentz transformation to the time axis

0-brane reduction: Adinkra

Example: Chiral Supermultipet

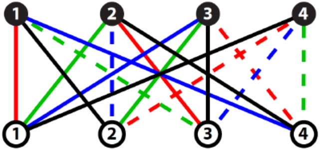
$$
\begin{array}{llll}
\mathrm{D}_1 A = i\,\Psi_1 & \mathrm{D}_2 A = i\,\Psi_2 & \mathrm{D}_3 A = i\,\Psi_3 & \mathrm{D}_4 A = i\,\Psi_4 \\
\mathrm{D}_1 B = -\,i\,\Psi_4 & \mathrm{D}_2 B = i\,\Psi_3 & \mathrm{D}_3 B = -i\,\Psi_2 & \mathrm{D}_4 B = i\,\Psi_1 \\
\mathrm{D}_1 F = i\,\partial_0 \Psi_2 & \mathrm{D}_2 F = -i\partial_0\,\Psi_1 & \mathrm{D}_3 F = -i\partial_0\,\Psi_4 & \mathrm{D}_4 F = i\partial_0\,\Psi_3 \\
\mathrm{D}_1 G = -\,i\partial_0 \Psi_3 & \mathrm{D}_2 G = -i\partial_0\,\Psi_4 & \mathrm{D}_3 G = i\partial_0\,\Psi_1 & \mathrm{D}_4 G = i\partial_0\,\Psi_2
\end{array}
$$

Figure 7: Transformation law after 0 brane reduction

Define

$$
\mathrm{D}_I \Phi_i \equiv i(L_I)_{i\hat{k}} \Psi_{\hat{k}} \qquad \mathrm{D}_I \Psi_{\hat{k}} \equiv (R_I)_{\hat{k}i} \partial_0 \Phi_i
$$

$$
(\mathrm{L}_1)_{i\hat{k}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}
$$



(a) Adinkra of (CS)

Projects currently working on

- Examination of Holoraumy Tensors for 4D, $\mathcal{N} = 1$ On-shell Supermultiplets (Matter-Gravitino, Supergravity)
- SU(3) anticommutators
- Deformed adinkras
- Casimir project

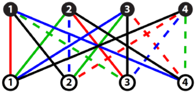
Project: Deformed Adinkras

Define

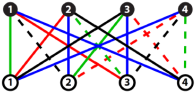
$$\mathrm{D}_I \Phi_i \equiv i(L_I)_{i\hat{k}} \Psi_{\hat{k}} \qquad \mathrm{D}_I \Psi_{\hat{k}} \equiv (R_I)_{\hat{k}i} \partial_0 \Phi_i$$

Supersymmetry algebra in 4D: $\{D_a, D_b\} = i2(\gamma^\mu)_{ab}\partial_\mu$
becomes Garden algebra after reduction:

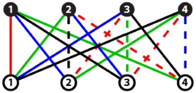
$$\begin{aligned} L_I R_J + L_J R_I &= 2\delta_{IJ} \\ R_J L_I + R_I L_J &= 2\delta_{IJ} \end{aligned} \qquad R_I = (L_I)^t$$



(a) Adinkra of (CS)



(b) Adinkra of (VS)



(c) Adinkra of (TS)

(1) Odd number of dashed lines. (2) 12 cycles in total.

Project: Deformed Adinkras

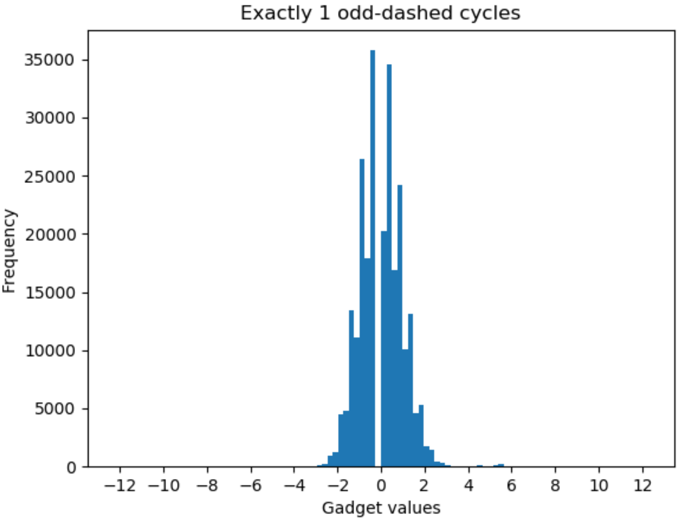
Define fermionic holoraumy matrices:
$\tilde{V}_{IJ} = \frac{1}{2}(L_I^t L_J - L_J^t L_I)$
Define a dot product (Gadget value) between different adinkras:
$G : \mathcal{A} \times \mathcal{A} \to Q : (R, R') \mapsto -\frac{1}{4}\sum_{I,J} Tr[(\tilde{V}_{IJ}^{(R)})\tilde{V}_{IJ}^{(R')})]$

Do dot product between any two adinkras (36,864 in total):

$\text{Gadget}_{(1)}$ Value	Count
- 1/3	127,401,984
0	1,132,462,080
1/3	84,934,656
1	14,155,776

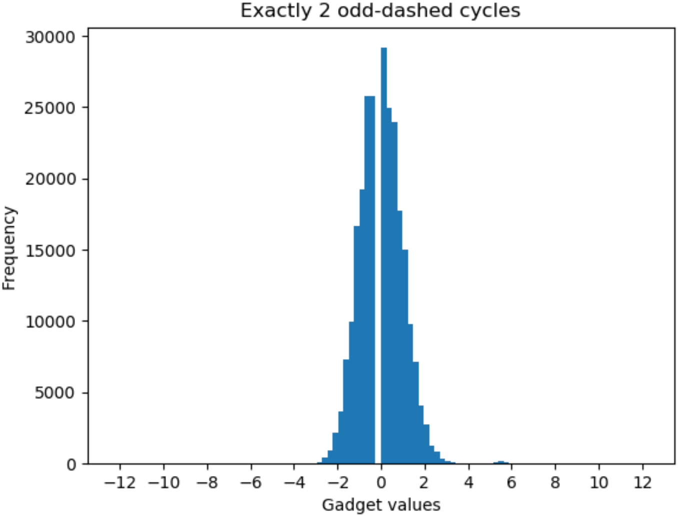
Project: Deformed Adinkras

1 odd-dashed cycles.png



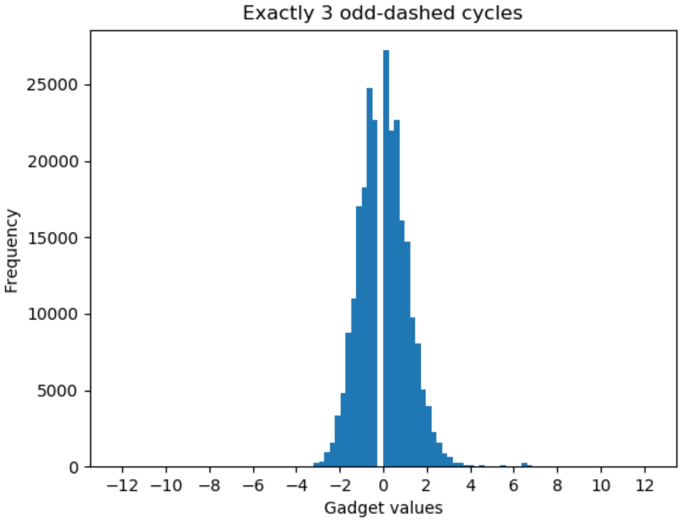
Project: Deformed Adinkras

2 odd-dashed cycles.png



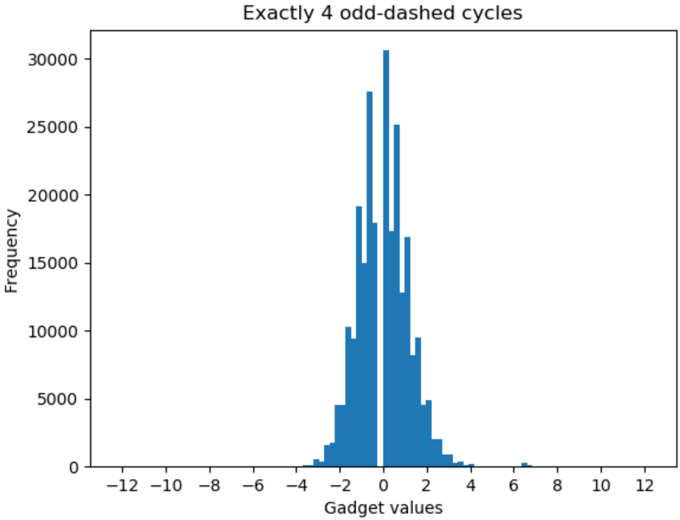
Project: Deformed Adinkras

3 odd-dashed cycles.png



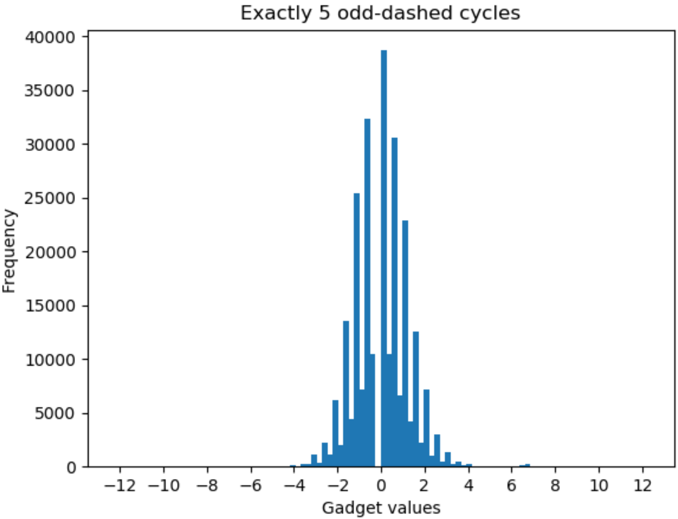
Project: Deformed Adinkras

4 odd-dashed cycles.png



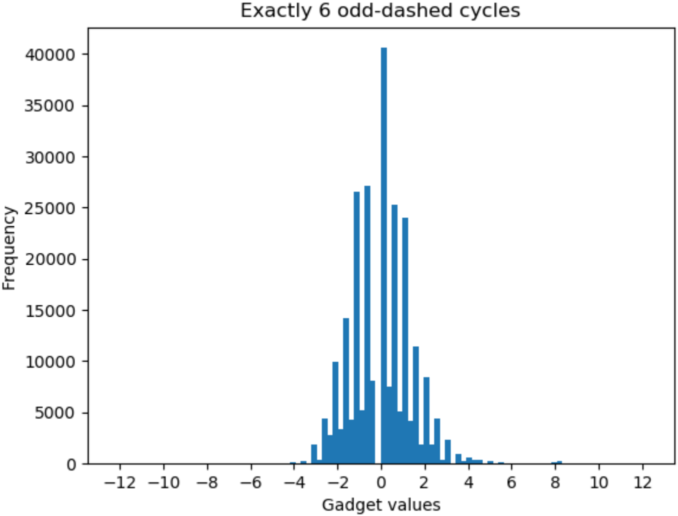
Project: Deformed Adinkras

5 odd-dashed cycles.png



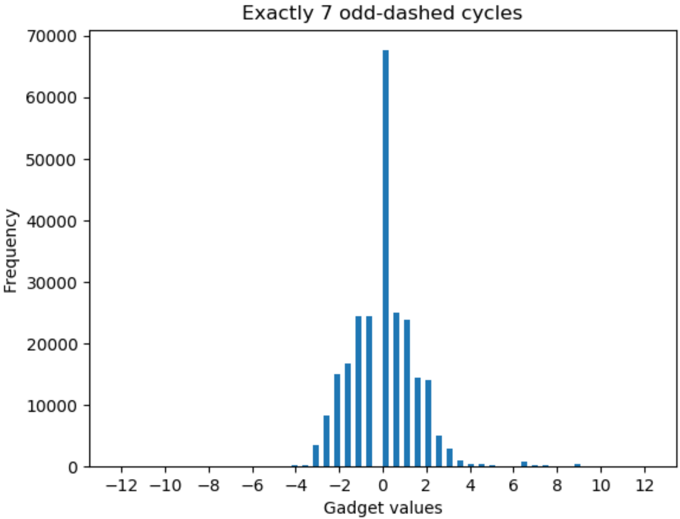
Project: Deformed Adinkras

6 odd-dashed cycles.png



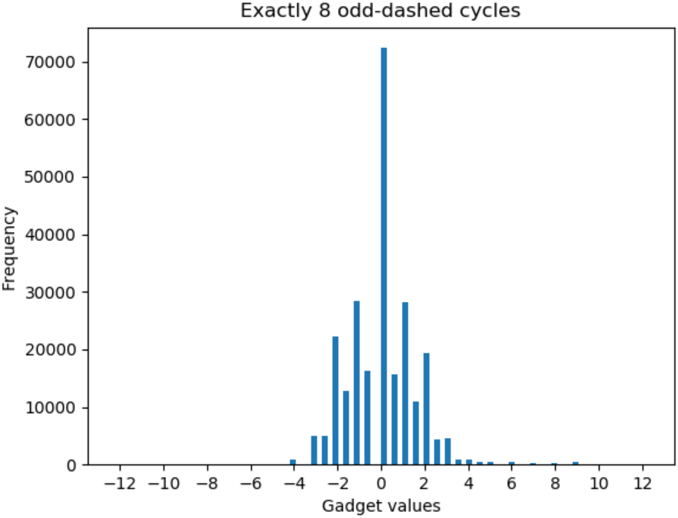
Project: Deformed Adinkras

7 odd-dashed cycles.png



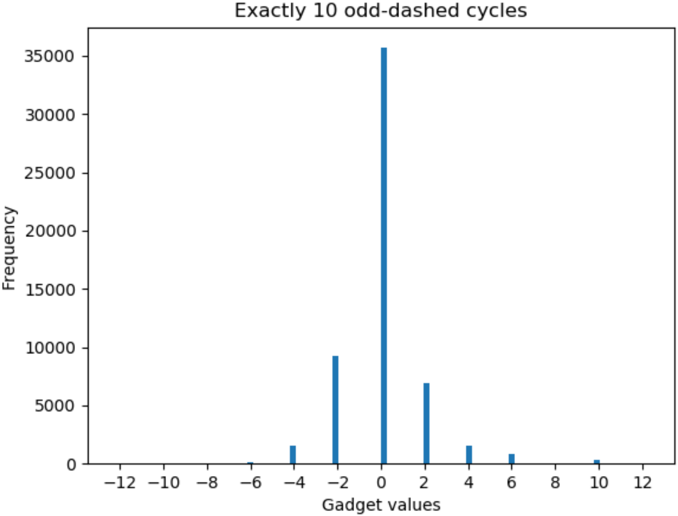
Project: Deformed Adinkras

8 odd-dashed cycles.png



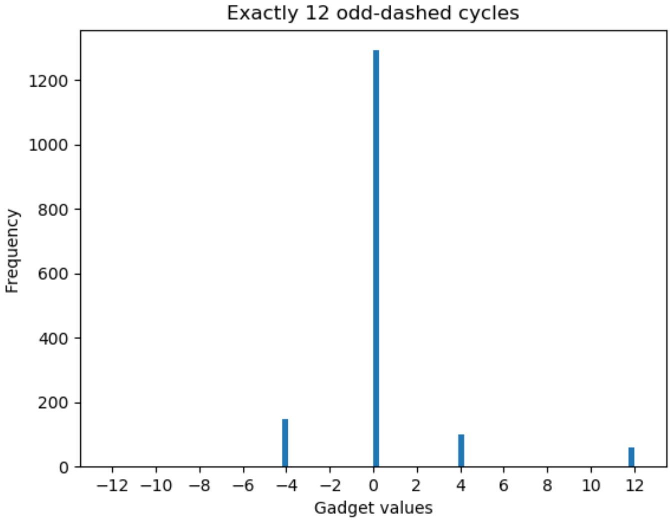
Project: Deformed Adinkras

10 odd-dashed cycles.png



Project: Deformed Adinkras

12 odd-dashed cycles.png



Project: Casimir project

su(3) algebra:

- Algebra:
  $[\lambda_i, \lambda_j] = if_{ijk}\lambda_k$
- Analogy of Holoraumy?:
  $\{\lambda_i, \lambda_j\}$
- Gadget value?
  $Tr(\{\lambda_i^{(R)}, \lambda_j^{(R)}\}\{\lambda_j^{(R')}, \lambda_i^{(R')}\})$

Supersymmetry algebra:

- Algebra:
  $\{D_a, D_b\} = i2(\gamma^\mu)_{ab}\partial_\mu$
- holoraumy tensor:
  $[D_a, D_b]\mathcal{F}_c \equiv [H^{\mu(R)}]_{abc}{}^d(\partial_\mu \mathcal{F}_d)$
- Gadget value:
  inner product between holoraumy tensors

Project: Casimir project

Gadgets:

$$
\begin{aligned}
G_1(R,R') &= Tr(\{\lambda_i^{(R)},\lambda_j^{(R)}\}\{\lambda_j^{(R')},\lambda_i^{(R')}\}) \\
&= \frac{1}{18}(p+1)(q+1)(p+q+2)\left(p^2+pq+3p+q^2+3q\right. \\
&\left.(4p^2+4pq+12p+4q^2+12q-9\right) \\
&= d\, C_2\,(4\,C_2-3)
\end{aligned}
$$

where d is the Weyl dimension formula:

$$d = \frac{1}{2}(p+1)(q+1)(p+q+2)$$

and

$$C_2 = \frac{1}{3}(p^2+q^2+pq+3p+3q)$$

is the eigenvalue of the quadratic casimir operator of su(3).

Project: Casimir project

Gadget value in su(3):

$$Tr(\{\lambda_i^{(R)},\lambda_j^{(R)}\}\{\lambda_j^{(R')},\lambda_i^{(R')}\})$$

$$G_1(R,R') = Tr(\{\lambda_i^{(R)},\lambda_j^{(R)}\}\{\lambda_i^{(R')},\lambda_j^{(R')}\})$$

$$G_2(R,R) = Tr(\{\lambda_i^{(R)},\lambda_j^{(R)}\}\{\lambda_j^{(R)},\lambda_k^{(R)}\}\{\lambda_k^{(R)},\lambda_i^{(R)}\})$$

$$G_3(R,R) = Tr(\{\lambda_i^{(R)},\lambda_j^{(R)}\}\{\lambda_j^{(R)},\lambda_k^{(R)}\}\{\lambda_k^{(R)},\lambda_l^{(R)}\}\{\lambda_l^{(R)},\lambda_i^{(R)}\})$$

$$G_4(R,R) = Tr(\{\lambda_i^{(R)},\lambda_j^{(R)}\}\{\lambda_j^{(R)},\lambda_k^{(R)}\}\{\lambda_k^{(R)},\lambda_l^{(R)}\}\{\lambda_l^{(R)},\lambda_p^{(R)}\}\{\lambda_p^{(R)},\lambda_i^{(R)}\})$$

$N(p,q)$	p	q	$G_1(R,R)$	$G_2(R,R)$	$G_3(R,R)$	$G_4(R,R)$	$G_5(R,R)$
1	0	0					
3	1	0	$\frac{28}{3}$	$\frac{53}{9}$	$\frac{371}{54}$	$\frac{548}{81}$	$\frac{1918}{243}$
6	2	0	$\frac{620}{3}$	$\frac{7225}{9}$	$\frac{238555}{54}$	$\frac{3534995}{162}$	$\frac{55836715}{486}$
8	1	1	216	702	3429	$\frac{28971}{2}$	$\frac{269109}{4}$
10	3	0	1260	11475	$\frac{244755}{2}$	$\frac{2549205}{2}$	$\frac{26814375}{2}$
15	2	1	$\frac{4400}{3}$	$\frac{102820}{9}$	$\frac{2885870}{27}$	$\frac{78479740}{81}$	$\frac{2164403930}{243}$
$15'$	4	0	$\frac{14420}{3}$	$\frac{680575}{9}$	$\frac{70507465}{54}$	$\frac{1810851910}{81}$	$\frac{93292782800}{243}$
21	5	0	$\frac{42280}{3}$	$\frac{3007550}{9}$	$\frac{227682385}{27}$	$\frac{17170978790}{81}$	$\frac{1296549205630}{243}$
24	3	1	$\frac{18200}{3}$	$\frac{750250}{9}$	$\frac{34391525}{27}$	$\frac{3118859825}{162}$	$\frac{283973161325}{972}$
27	2	2	6264	81918	1197531	17297982	250983144

Project: Casimir project

Gadget value in su(3):
$Tr(\{\lambda_i^{(R)}, \lambda_j^{(R)}\}\{\lambda_j^{(R')}, \lambda_i^{(R')}\})$

$$G_1(R, R') = Tr(\{\lambda_i^{(R)}, \lambda_j^{(R)}\}\{\lambda_i^{(R')}, \lambda_j^{(R')}\})$$

$$G_2(R, R) = Tr(\{\lambda_i^{(R)}, \lambda_j^{(R)}\}\{\lambda_j^{(R)}, \lambda_k^{(R)}\}\{\lambda_k^{(R)}, \lambda_i^{(R)}\})$$

$$G_3(R, R) = Tr(\{\lambda_i^{(R)}, \lambda_j^{(R)}\}\{\lambda_j^{(R)}, \lambda_k^{(R)}\}\{\lambda_k^{(R)}, \lambda_l^{(R)}\}\{\lambda_l^{(R)}, \lambda_i^{(R)}\})$$

$$G_4(R, R) = Tr(\{\lambda_i^{(R)}, \lambda_j^{(R)}\}\{\lambda_j^{(R)}, \lambda_k^{(R)}\}\{\lambda_k^{(R)}, \lambda_l^{(R)}\}\{\lambda_l^{(R)}, \lambda_p^{(R)}\}\{\lambda_p^{(R)}, \lambda_i^{(R)}\})$$

We also found that G_2, G_3, G_4 are all poportional to the polynomial of C_2. Starting from G5, no more polynomial of C_2 can be found.

$$\hat{C}_2 = \lambda_i \lambda_i$$

$$\hat{C}_3 = d_{ijk} \lambda_i \lambda_j \lambda_k$$

Analytically derivation:
$\lambda_i ... \lambda_j ... \lambda_j ... \lambda_i ... \propto$ Identity matrix, using commutator relation to swap the order.

What to do next

- Check the fermonic holoraumy matrices of MGM and SG supermultiplet belong to what group.
- Investigate the relation between the gadget value and the dashed cycle condition.
- Find the explicit formula for the eigenvalues of the possible "new" casimir operator.

Reference

- S.N.Mak, S.J.Gates, "Finding the Roots of Supersymmetry", poster.
- "On the Four Dimensional Holoraumy of the 4D, N = 1 Complex Linear Supermultiplet". International Journal of Modern Physics AVol. 33, No. 12, 1850072 (2018).
- "Adinkras from ordered quartets of BC4 Coxeter group elements and regarding 1,358,954,496 matrix elements of the Gadget". High Energ. Phys. (2017).

Thank you

Q & A