SURE report: Exploring the relation between Lie algebra, Supersymmetry, and Adindra

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Abstract. The holoraumy tensor has been found to play a role of classifing supersymmetry algebra similar to the role of eigenvectors and eigenvalues and the corresponding weight space in su(3) lie algebra. Further investigation of holoraumy tensor in on-shell 4D, N=1 supermultiplets and the counterpart in su(3) was carried out. High relevance of Garden algebra and gadget value distribution was also found.

1 Introduction

1.1 Weight Space of SU(3) Algebra

Weight space is crucial in classifying different representation in Lie algebra.

For example, in the fundamental representation of su(3) algebra, the 8 generator, also known as the Gell-Mann matrices (p=1, q=0) are given by:

$$
\begin{aligned}
&2\lambda_1 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix} \quad 2\lambda_2 = \begin{pmatrix} 0&-i&0\\i&0&0\\0&0&0 \end{pmatrix} \quad 2\lambda_3 = \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix}\\
&2\lambda_4 = \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix} \quad 2\lambda_5 = \begin{pmatrix} 0&0&-i\\0&0&0\\i&0&0 \end{pmatrix}\\
&2\lambda_6 = \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix} \quad 2\lambda_7 = \begin{pmatrix} 0&0&0\\0&0&-i\\0&i&0 \end{pmatrix} \quad 2\lambda_8 = \tfrac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&-2 \end{pmatrix}
\end{aligned}
\tag{1}
$$

The algebra is given by:

$$[(\lambda_i), (\lambda_j)] = i f_{ijk}(\lambda_k) \tag{2}$$

It is easy to verify that λ_3 and λ_8 commute with each other. Therefore three eigenvectors $|u\rangle$, $|d\rangle$, $|s\rangle$ of both λ_3 and λ_8 can be found:

$$
\begin{aligned}
(\lambda_3)\,|u\rangle &= \frac{1}{2}\,|u\rangle\\
(\lambda_8)\,|u\rangle &= \frac{1}{2\sqrt{3}}\,|u\rangle
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
(\lambda_3)\,|d\rangle &= -\frac{1}{2}\,|d\rangle \\
(\lambda_8)\,|d\rangle &= \frac{1}{2\sqrt{3}}\,|d\rangle
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
(\lambda_3)\,|s\rangle &= 0\,|s\rangle \\
(\lambda_8)\,|s\rangle &= \frac{1}{\sqrt{3}}\,|s\rangle
\end{aligned}
\tag{5}
$$

Three pairs of eigenvalues can be represented in a coordinate and thus obtain the weight space diagram (see Fig. 1):

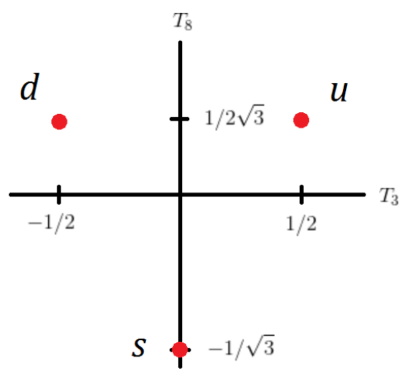


Fig. 1. Weight space for p=1, q=0

Weight space are different for different value of p and q, thereby different representation.

1.2 Supersymmetry Algebra

Creating Gamma matrices from pauli matrices:

$$
\begin{aligned}
\gamma^0 &= i(\sigma^3 \otimes \sigma^2) \qquad & \gamma^1 &= (\mathbb{I}_2 \otimes \sigma^1) \\
\gamma^2 &= (\sigma^2 \otimes \sigma^2) \qquad & \gamma^3 &= (\mathbb{I}_2 \otimes \sigma^3)
\end{aligned}
\tag{6}
$$

Then, the 4D, N=1 Chiral supermultiplet is given by [1]:

$$
\begin{aligned}
\mathrm{D}_a A =& \psi_a \\
\mathrm{D}_a B =& i(\gamma^5)_a{}^b \psi_b \\
\mathrm{D}_a \psi_b =& i(\gamma^\mu)_{ab}\partial_\mu A - (\gamma^5\gamma^\mu)_{ab}\partial_\mu B - iC_{ab}F + (\gamma^5)_{ab}G \\
\mathrm{D}_a F =& (\gamma^\mu)_a{}^b \partial_\mu \psi_b \\
\mathrm{D}_a G =& i(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \psi_b
\end{aligned}
\tag{7}
$$

Supersymmetry algebra is given by:

$$
\{D_a, D_b\}\psi_c = i2(\gamma^\mu)_{ab}\partial_\mu \psi_c \tag{8}
$$

We want to do the same thing to different supermultiplets just as weight space classifying different representation of su(3) algebra. However the supercovariant derivative D_a transform bosons to fermions or fermions to bosons. Therefore it seems that there is no eigenvectors can be found and the method in su(3) can not be applied to the SUSY case.

One approach of tackling this problem is to calculate the commutator of covariant derivative acting on fermion fields.

$$[D_a, D_b]\mathcal{F}_c \equiv [\boldsymbol{H}^{\mu(R)}]_{abc}{}^{d}(\partial_\mu \mathcal{F}_d)\boldsymbol{d} \tag{9}$$

Previous research has shown that the for the 4D, N=1 minimal supermultipets [2]:

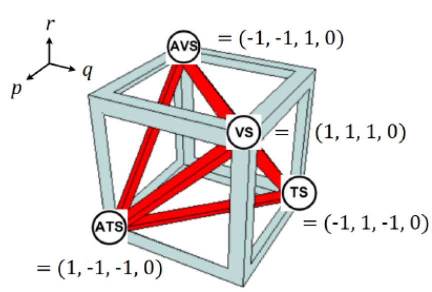
$$\begin{aligned}\left[\boldsymbol{H}^{\mu}(p_{(R)}, q_{(R)}, r_{(R)}, s_{(R)})\right]_{abc}{}^{d} =& -i2\big[p_{(R)}C_{ab}(\gamma^\mu)_c{}^d + q_{(R)}(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d \\ &+ r_{(R)}(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d + \frac{1}{2}s_{(R)}(\gamma^5\gamma^\nu)_{ab}(\gamma^5[\gamma_\nu, \gamma^\mu])_c{}^d\big]\end{aligned} \tag{10}$$

The result is given by

$(\widehat{\mathcal{R}})$	$p_{(\mathcal{R})}$	$q_{(\mathcal{R})}$	$r_{(\mathcal{R})}$	$s_{(\mathcal{R})}$
(CS)	0	0	0	1
(VS)	1	1	1	0
(TS)	- 1	1	- 1	0

Fig. 2. Value of $p_{(R)}$, $q_{(R)}$, $r_{(R)}$, $s_{(R)}$

Therefore some constants is obtained from calculating the holoraumy tensors which are intrinsic for each supermultiplet.



$$(CS) = (0, 0, 0, 1)$$

Fig. 3. Tetrahedron in the 3D subspace

Taking the fist three parameters p, q, r, the four supermultiplets: (VS), (TS), (AVS), and (ATS) happens to be the vertices of a tetrahedron if place (p,q,r) in a coordinate (Fig. 3).

1.3 Adinkra and Gadget Value

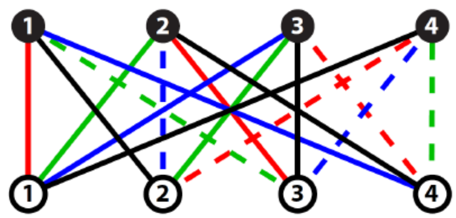
One can do a lorentz transform so that all the fields only have time dependence. This process is called "0-brane reduction".

The transfermation law of the 4D, N=1 minimal supermultiplet after the reduction can be writen in the form [1]:

$$\begin{aligned} D_I\Phi_i &= i(L_1)_{i\hat{k}}\Psi_{\hat{k}} \\ D_I\Psi_{\hat{k}} &= i(L_1)_{\hat{k}i}\Phi_i \end{aligned} \tag{11}$$

Where L and R are 4 by 4 signed permutation matrices.

Considering L and R matrices as the adjacency matrices, a graph can be generated:



(a) Adinkra of (CS)

Fig. 4. Adinkra of Chiral multiplet

Where white nodes and black nodes represent bosons and fermions respectively. Solid lines represent +1 and dashed lines represent −1 of the element from L and R matrices. The supersymmetry algebra after 0-brane reduction is giving by the Garden algebra [1]:

$$\begin{aligned} L_IR_J + L_JR_I &= 2\delta_{IJ}I_{4\times4} \\ R_IL_J + R_JL_I &= 2\delta_{IJ}I_{4\times4} \end{aligned} \tag{12}$$

and

$$R_I = (L_I)^t \tag{13}$$

- The Garden algebra implies for a cycle of two colors in the Adinkra, there will always be odd number of dashed lines.
- The set of L and R matrices also happens to be complete in the sense that adding the fours matrices together forms a matrices without having zero element.

These two requirement defines a valid adinkra, and there are 36863 four colors adinkras in total.

The holoraumy matrix is defined to be:

$$\tilde{V}_{IJ} = (L_I)^tL_J + (L_J)^tL_I \tag{14}$$

and a dot product between two adinkras, which is called the gadget value, is defined to be [3]:

$$G : \mathcal{A}\times\mathcal{A} \to Q : (R, R') \mapsto -\frac{1}{4}\sum_{I,J} Tr[(\tilde{V}_{IJ}^{(R)})\tilde{V}_{IJ}^{(R')})] \tag{15}$$

However, it was found that only four different gadget values: $-\frac{1}{3}$, 0, $\frac{1}{3}$, 1 was found when doing dot product between any two adinkras.

2 Research Projects

Four research projects are on-going and are briefly introduced below.

2.1 Examination the Holoraumy Tensors of 4D, N=1 On-Shell Supermultiplets

On-shell supermultiplets does not include auxiliary fields that don't affect the equation of motion.

The supermultiplet under investigation are:

- Chiral and Complex Linear supermultiplet
- Vector supermultiplet
- Axial Vector supermultiplet
- Matter-Gravitino Matter
- Supergravity supermultiplets

The objective of these research is to calculate the algebra and the holoraumy tensor of these supermultiplets. Basically the anticommutator $\{D_a, D_b\}$ and commutator $[D_a, D_b]$ of the supercovariant derivative.

Calculations was based on expanding the result on a set of basis:

Symmetric basis:

$$\{(\gamma^\mu)_{ab},\ \ ([\gamma^\mu, \gamma^\nu])_{ab}\}$$

Antisymmetric basis:

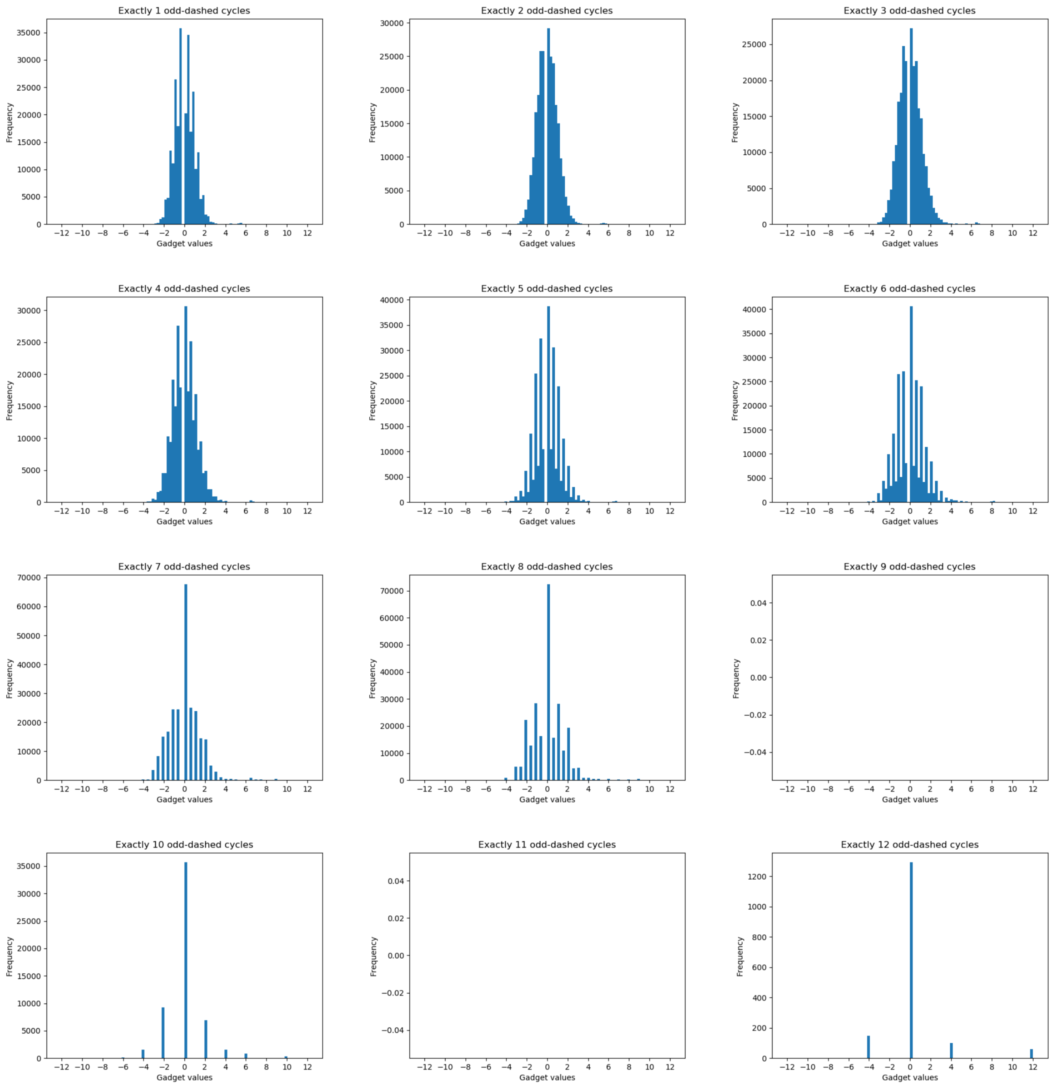
$$\{C_{ab},\ \ (\gamma^5)_{ab},\ \ (\gamma^5\gamma^\mu)_{ab}\}$$

One of the result from the calculation is that the L and R matrices after 0-brane reduction are not the transpose of each other, and therefore cannot be represented by Adinkra.

2.2 Deformed Adinkra

We examined the relation between completeness condition (the second requirement for a adinkra) and the gadget value distribution by taking large sample of quartets of signed permutation matrices.

Strong relevance of completeness and gadget value diversity are found:



2.3 Decomposition of Anticommutator of su(3) Generators

Young tableau has been an important tool of decomposing the outer product of two su(3) representations into the direct sum of multiple irreducible representations.

One way of interpret this result is that given a representation, when considering the vector space of all the matrices of that size, there should be a set of basis consists of matrices in which different representations are embedded.

Take the fundamental representation of su(3) as an example, all 3 by 3 matrices can be expressed as the linear combination of the 8 generators and the identity matrix. Which can be obtained simply be doing the anticommutator of the generators:

$$\{\boldsymbol{\lambda}_a, \boldsymbol{\lambda}_b\} = \frac{1}{3}\delta_{ab}\boldsymbol{I}_{3\times 3} + d_{abc}\boldsymbol{\lambda}_c \tag{16}$$

where $\boldsymbol{\lambda}$'s are the generators of the fundamental representation and d_{abc} is the anomaly coefficient.

We are tring to do the same case for larger representations where things cannot be solved by simply calculating the anticommutator of the generators.

2.4 Casimir Project

Remember that the dot product between adinkra of different representations is given by the gadget formula (Equation 15), where the holoraumy matrix $\tilde{V}$ is the supersymmetry algebra after the 0-brane reduction.

Similarly "the Gadget" between two representations of su(3) algebra R and R' can be constructed in following way [4]:

$$C_{4,g}(R,R') \propto \sum_{i=1,j=1}^{8} Tr(\{\boldsymbol{\lambda}_i^{(R)}, \boldsymbol{\lambda}_j^{(R)}\}\{\boldsymbol{\lambda}^{j(R')}, \boldsymbol{\lambda}^{i(R')}\}) \tag{17}$$

We first examined "the Gadget" from two identical representations. The result obtained from both computation and analytically calculation shows that:

$$\begin{aligned} G_{4,g}(R,R) &= \frac{1}{18}(p+1)(q+1)(p+q+2)(p^2+pq+3p+q^2+3q)(4p^2+4pq+12p+4q^2+12q-9) \\ &= d\,C_2\,(4\,C_2-3) \end{aligned} \tag{18}$$

where d is the Weyl dimension formula:

$$d = \frac{1}{2}(p+1)(q+1)(p+q+2) \tag{19}$$

and

$$C_2 = \frac{1}{3}(p^2+q^2+pq+3p+3q) \tag{20}$$

is the eigenvalue of the quadratic casimir operator of su(3).

Higher order of "the Gadget" was also considered. For example:

$$C_{6,g}(R,R) \propto \sum_{i=1,j=1,k=1}^{8} Tr(\{\boldsymbol{\lambda}_i^{(R)}, \boldsymbol{\lambda}_j^{(R)}\}\{\boldsymbol{\lambda}^{j(R)}, \boldsymbol{\lambda}_k^{(R)}\}\{\boldsymbol{\lambda}^{k(R)}, \boldsymbol{\lambda}^{i(R)}\}) \tag{21}$$

We we further raise the order of "the Gadget". Some term that cannot be constructed by polynomial of quadratic casimir operator appeared and requires further investigation.

3 Summary

Many object investigated in these project are new in the sense that we really know little about it. The objectives of these project were to make analogy between su(3) algebra and supersymmetry algebra. Anticommutators in su(3) algebra and commutators in supersymmetry algebra have been considered as the route of constructing objects that are intrinsic to each representation. Many rather concise results were obtained during the process and motivate further investigation.

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